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ALGEBRAS GROUPS AND GEOMETRIES

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**CATEGORY OF CAYLEY-KLEIN GROUPS AND
FUNCTOR CATEGORY OF VKS-TREES , 1**

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FUNCTOR CATEGORY OF VKS-TREES¹**

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Abstract

Following Manin, Leinster, Markl, Aguiar and Sottile we review definitions, and basic properties of operads, and trees, and algebras over these structures in Sections 1–13. It is intended to consider categories of operads and trees as a whole, the content of each Section taking into account the contents of other ones. But also the same problem of categories of operads and trees may be discussed repeatedly, however from different points of view.

The next Sections we begin by introducing the method of Vilenkin-Kuznetsov-Smorodinsky (VKS)-trees. The Cayley-Klein groups are constructed on parameters, each of which can be real, purely imaginary and Clifford dual one. Then representations of orthogonal Cayley-Klein groups are constructed with the method of VKS-tree. Finally, a Cayley-Klein category is defined and the functor category of VKS-trees is constructed.

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1 Operads as a Generalization of Associative Algebras

1.1 Classical linear operads

In Sections 1–2 and 12–13 we fix a ground field k of characteristic zero and denote by $VECT$ the category of linear spaces over k . All tensor products are taken over k unless it is explicitly stated otherwise. The symmetric group $\mathbf{S}_n$ is defined as the group of the bijections $\underline{n} \rightarrow \underline{n}$ where $\underline{n} = \{1, \dots, n\}$.

Classical linear algebra deals with a linear space V endowed with a family $\mathcal{O}$ of linear operators $V \rightarrow V$. Usually it is convenient to close $\mathcal{O}$ by adding all operator compositions and their linear combinations to $\mathcal{O}$. In this way linear algebra becomes the study of associative k -algebras and their linear representations.

Classical linear operads arise in the same way when we start with a linear space V endowed with a family $\mathcal{P}$ of *polylinear* operators $V^{\otimes m} \rightarrow V, m = 1, 2, 3, \dots$ (for example, an associative algebra is such a space endowed with a multiplication map $V^{\otimes 2} \rightarrow V$). Closing $\mathcal{P}$ with respect to compositions (of functions with many variables) and linear combinations we get a (concrete) classical linear operad together with its linear representation in V . Axiomatizing the universal properties of such an object, we get the following notion.

DEFINITION 1.1 A **classical linear operad** $\mathcal{P}$ consists of the data a) – d) satisfying the axioms A) – C) below.

- a) A family of linear spaces $\mathcal{P}(l)$, for all $l \geq 1$.
- b) A left/right linear action of $\mathbf{S}_l$ on $\mathcal{P}(l)$, for all $l \geq 1 : s \in \mathbf{S}_l$ maps $f \in \mathcal{P}(l)$ to $fs = s^{-1}f$.
- c) A family of composition maps $\gamma(k_1, \dots, k_l)$, for all $l \geq 1, k_a \geq 1$:

$$\gamma(k_1, \dots, k_l) : \mathcal{P}(l) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_l) \rightarrow \mathcal{P}(k_1 + \dots + k_l). \quad (1.1)$$

- d) (Optional). An identity element $I \in \mathcal{P}(1)$.

We will state the axioms for these data in two forms: directly in terms of γ and in functional notation. For the latter, put $\underline{\mathcal{P}} = \oplus_{k=1}^{\infty} \mathcal{P}(k)$ and notice that (1.1) allows us to consider each $f \in \mathcal{P}(l)$ as a polylinear function $\underline{\mathcal{P}}^l \rightarrow \underline{\mathcal{P}}$:

$$f(g_1, \dots, g_l) := \gamma(f \otimes g_1 \otimes \dots \otimes g_l), \quad (1.2)$$

where $\gamma = \gamma((k_1, \dots, k_l)$ if $g_a \in \mathcal{P}(k_a)$. We will often write simply γ for such multigraded components of the operadic composition.

A) *The symmetric group $\mathbf{S}_l$ acts on the functions (represented by) $\mathcal{P}(l)$ by permutation of arguments:*

$$(fs)(g_1, \dots, g_l) = f(s(g_1, \dots, g_l)). \quad (1.3)$$

In γ -notation:

$$\gamma(fs \otimes g_1 \otimes \dots \otimes g_l) = \gamma(f \otimes s(g_1 \otimes \dots \otimes g_l)). \quad (1.4)$$

In addition, for $s_1 \in S_{k_1}, \dots, s_l \in S_{k_l}$, denote by $s_1 \times \dots \times s_l \in S_{k_1 + \dots + k_l}$ the image of $(s_1, \dots, s_l)$ acting blockwise upon

$$(1, \dots, k_1 | k_1 + 1, \dots, k_1 + k_2 | \dots | k_1 + \dots + k_{l-1} + 1, \dots, k_1 + \dots + k_l).$$

Then

$$f(g_1 s_1, \dots, g_l s_l) = (f(g_1, \dots, g_l))(s_1 \times \dots \times s_l). \quad (1.5)$$

In γ -notation:

$$\gamma(f \otimes g_1 s_1 \otimes \dots \otimes g_l s_l) = (\gamma(f \otimes g_1 \otimes \dots \otimes g_l))(s_1 \times \dots \times s_l). \quad (1.6)$$

B) *The composition maps are associative with respect to the substitution (in functional notation). That is, for any $f \in \mathcal{P}(l)$, $g_a \in \mathcal{P}(k_a)$, $a = 1, \dots, l$, and $h_{a,b} \in \mathcal{P}(l_{a,b})$, $b = 1, \dots, k_a$, we have*

$$\begin{aligned} [f(g_1, \dots, g_l)](h_{1,1}, \dots, h_{1,k_1}; \dots; h_{l,1}, \dots, h_{l,k_l}) = \\ = f(g_1(h_{1,1}, \dots, h_{1,k_1}), \dots, g_l(h_{l,1}, \dots, h_{l,k_l})). \end{aligned} \quad (1.7)$$

In γ -notation:

$$\begin{aligned} \gamma(\gamma(f \otimes g_1 \otimes \dots \otimes g_l) \otimes h_{1,1} \otimes \dots \otimes h_{l,k_l}) = \\ = \gamma(f \otimes \gamma(g_1 \otimes h_{1,1} \otimes \dots \otimes h_{1,k_1}) \otimes \dots \otimes \gamma(g_l \otimes h_{l,1} \otimes \dots \otimes h_{l,k_l})). \end{aligned} \quad (1.8)$$

C) (Optional). If $\mathcal{P}$ is endowed with identity $I \in \mathcal{P}(1)$, then I (resp. $I^{\otimes n}$) become left (resp. right) identical functions:

$$I(g) = g; \quad f(I, \dots, I) = f, \quad (1.9)$$

$$\gamma(I \otimes g) = g; \quad \gamma(f \otimes I \otimes \dots \otimes I) = f. \quad (1.10)$$

An operad endowed with identity which is considered as a part of its structure will be called a **unital operad**.

We will often call the classical linear operads simply operads until the introduction of other versions of this notion.

EXAMPLE 1.1 Let (E, μ) be an associative algebra with multiplication $\mu : E \otimes E \rightarrow E$. Define an operad $\mathcal{P}_E$ by $\mathcal{P}_E(1) = E$, $\mathcal{P}_E(l) = \{0\}$ for $l \geq 2$, $\gamma(1) = \mu$, the rest of the data being self-explanatory. Operadic associativity of γ is clearly equivalent to the associativity of μ .

Conversely, for any operad $\mathcal{P}, \mathcal{P}(1)$ with multiplication $\gamma(1)$ is an associative algebra. Operadic identity becomes algebra identity and vice versa.

EXAMPLE 1.2 Let V be a linear space. Define the operad $OpEnd(V)$ by the following data:

$$OpEnd(V)(l) = Hom_{VECT}(V^{\otimes l}, V), \quad (1.11)$$

S_l acts by permuting arguments as in (1.3), the composition γ is defined by substitution as in the left-hand side of (1.2), and $I = id_V$.

DEFINITION 1.2 A **morphism** of operads $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$ is a family of linear maps $\varphi(l) : \mathcal{P}(l) \rightarrow \mathcal{Q}(l)$, $l \geq 1$, compatible with the action of symmetric groups, composition, and optionally, mapping $I_{\mathcal{P}}$ to $I_{\mathcal{Q}}$.

Thus we have defined a category of classical linear operads *OPER*. In fact, we allow some ambiguity, because the existence of the identity is optional, and, even if it exists, we may decide not to consider it as a part of the structure when we define morphisms. This extends the common ambiguity in the definition of the category of associative algebras.

REMARK 1.1 Denote by *ASS* one of the two categories of associative k -algebras (with or without identity). Constructions of Example 1.1 extend to the functors $ASS \rightarrow OPER$ and $OPER \rightarrow ASS$ which are adjoint to each other from both sides so that we have canonical identifications

$$\begin{aligned} Hom_{OPER}(\mathcal{P}, \mathcal{P}_A) &= Hom_{ASS}(\mathcal{P}(1), A), \\ Hom_{OPER}(\mathcal{P}_A, \mathcal{P}) &= Hom_{ASS}(A, \mathcal{P}(1)) . \end{aligned}$$

In particular, *ASS* is a full subcategory of *OPER*.

1.2 Operads as classifiers of algebras of different species

By species we mean here a general notion whose specializations include, e.g., associative, Cayley-Klein and Lie, commutative, and Poisson algebras; cf. Subsection 1.3 below.

DEFINITION 1.3 Let $\mathcal{P}$ be an operad and V a linear space. A structure of $\mathcal{P}$ -algebra on V , or equivalently, a **linear representation** of $\mathcal{P}$ in V , is a morphism of operads $\rho : \mathcal{P} \rightarrow OpEnd(V)$ sending I to id_V if $\mathcal{P}$ is unital.

As Definition 1.1 shows, $\underline{\mathcal{P}} = \bigoplus_{l \geq 1} \mathcal{P}(l)$ has a canonical structure of $\mathcal{P}$ -algebra (regular representation).

Generally, to define a structure of $\mathcal{P}$ -algebra on V is the same as to define for every element $f \in \mathcal{P}(l)$ a l -ary multiplication map $m_f : V^{\otimes l} \rightarrow V$ linearly depending on f , translating γ -composition to substitution and the action of the symmetric groups to the permutation of the arguments.

DEFINITION 1.4 Let V, W be two $\mathcal{P}$ -algebras. A morphism between them is a linear map $\varphi : V \rightarrow W$ such that for every $f \in \mathcal{P}(l)$ we have

$$\varphi(m_l^V(v_1 \otimes \cdots \otimes v_l)) = m_f^W(\varphi(v_1) \otimes \cdots \otimes \varphi(v_l)) . \quad (1.12)$$

We will show that for certain species C of k -algebras which we may call “operadic” one can find a unital operad COp such that COp -algebras and morphisms between them “are” algebras of the species C and their morphisms.

REMARK 1.2 *Let us start with an example, taking again associative algebras without unit, this time considered as a species. Besides the identity map $V \rightarrow V$, any associative algebra is commonly given by one generating bilinear multiplication $m : V \otimes V \rightarrow V$, but the transposition of arguments transforms it into another multiplication m^{op} . Therefore we must put $AssOp(1) = \langle I \rangle$ (brackets denoting the linear span), $AssOp(2) = \langle m, m^{op} \rangle$, the regular representation of S_2 . In $AssOp(3)$ we have then twelve ternary operations that can be constructed from I, m, m^{op} : in the functional notation they are $m(m, I)$, $m(I, m^{op})$, etc. In plain words, each such operation applied to $v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$ picks two v_i 's, multiplies them in some order, and then multiplies the result by the remaining v_l .*

These twelve ternary operations are related by identities expressing the associativity of m and its consequence, that of m^{op} : $m(m, I) = m(I, m)$, etc. As a result, $AssOp(3)$ is isomorphic to the regular representation of S_3 generated by, say, $m(m, I)$.

The general pattern is as follows. Pick an infinite sequence of independent non-commuting but associative variables $x_1, x_2, x_3, \dots$. Instead of $m, m^{op}, m(I, m^{op})$, etc., write the value of the respective operation applied to the initial segment of this sequence, getting respectively $x_1x_2, x_2x_1, x_1x_3x_2$, etc. A contemplation shows that one can thus identify $AssOp(n)$ with the linear space generated by all associative monomials $x_{s(1)} \dots x_{s(n)}$ where $s \in \mathbf{S}_n$, with the evident action of $\mathbf{S}_n$.

Namely, $m(\dots(m(m, I), I) \dots)$ produces the monomial $(\dots((x_1x_2)x_3) \dots)x_n = x_1 \dots x_n$, and the application of $\mathbf{S}_n$ furnishes the rest. It remains to describe the γ -composition of a monomial $x_{s(1)} \dots x_{s(n)}$ with $g_1 \otimes \dots \otimes g_n \in \bigoplus_{a=1}^n AssOp(l_a)$. We first replace arguments $x_1, \dots, x_{l_a}$ in g_a by adding $l_1 + \dots + l_{a-1}$ to all subscripts thus getting $\tilde{g}_a$, and then put

$$\gamma(x_{s(1)} \otimes \dots \otimes x_{s(n)} \otimes g_1 \otimes \dots \otimes g_n) := \tilde{g}_{s(1)} \dots \tilde{g}_{s(n)} .$$

Now let us try to construct a functor from $AssOp$ -algebras to associative algebras.

A structure of an $AssOp$ -algebra on V , clearly, is uniquely determined by the restriction of the operadic morphism

$$\rho(2) : AssOp(2) \rightarrow Hom_{VECT}(V^{\otimes 2}, V) .$$

However, the image of $\rho(2)$ is a two-dimensional space of multiplications $\{am + bm^{op}\}$ whereas classically we need just one associative multiplication. Let us write

the associativity equation $\mu(\mu, I) = \mu(I, \mu)$ for $\mu = am + am^{op}$ in the functional notation with free arguments x, y, z :

$$\begin{aligned}\mu(\mu, I) &= a[(axy + byx)z] + b[z(axy + byx)] , \\ \mu(I, \mu) &= a[x(ayz + bzy) + b[(ayz + bzy)x] .\end{aligned}$$

Comparing coefficients, one sees that the universal associativity (in any linear representation) is equivalent to $ab = 0$. Hence the best we can do is to pinpoint in any *AssOp*-algebra V two lines of associative multiplications: $\langle \rho(m) \rangle$ and $\langle \rho(m^{op}) \rangle$. An additional choice of unit would reduce each line to one (non-zero) element, however there is nothing in the structure of *AssOp* that would help us to do this. In fact, we encounter here a general problem: how to account for eventual structural special elements, i.e. 0-ary operations. In principle, we could have extended the definition of an operad $\mathcal{P}$ by including $\mathcal{P}(0)$ and extending correspondingly (1.1). In particular, we can put $\text{AssOp}(0) = \text{ground field}$, $\text{OpEnd}(V)(0) = V$, and define the identity in V as the image of 1. In other cases this might not work.

We will now summarize the preceding discussion in a deliberately vague “meta-theorem” (for more precise statements, see below).

1.3 Species of algebras and operads

Let C be a category of algebras which is defined by a family of multilinear operations $\{m_i | i \in I\}$ and a family of universal identities between them constructed of compositions and linear combinations. Morphisms in C are linear maps compatible with m_i ’s. Examples: associative algebras without identity (multiplication; associativity); Cayley-Klein and Lie algebras (bracket; skew-symmetry; Jacobi identity); Poisson algebras without identity (multiplication, bracket; associativity, commutativity, Jacobi, Leibniz); commutative rings with an m -dimensional linear space of pairwise commuting derivations, etc.

Then one can construct a classical linear operad COp with the following properties.

a) $COp(l)$ as a representation space of $\mathbf{S}_l$ is isomorphic to a subspace of the free algebra $F_C(x_1, \dots, x_l)$ of the species C freely generated by l independent variables $x_1, \dots, x_l$. This subspace consists of forms of total degree l linear in each x_a , upon which $\mathbf{S}_l$ acts by permuting arguments.

b) Compositions γ are induced by substitution.

c) To give a structure of a COp -algebra on a space V is the same as giving a set of structures of species C on V . Various elements of this set are obtained by choosing in COp various generating families of solutions $\{m'_i\}$ of the universal identities defining C . The group $Aut(COp)$ acts transitively on this set.

d) The category of COp -algebras is equivalent to the category of algebras of the species C . Every choice of generators $\{m_i\}$, as above fixes one equivalence functor. However, two different choices may lead to non-isomorphic functors. This happens, e.g., with $AssOp$ and functors corresponding to m and m^{op} .

To give a precise statement and proof of this theorem, we would have to explain in more detail the two different notions of “freeness” and “defining an object by generators and relations”: separately for operads and algebras over a given operad. Above we used them on an intuitive level.

Before proceeding further, we want to list the limitations of the operadic approach to species, some of which can be overcome by modifying the notion of the classical operad.

- We cannot account for the structure constants, partly because of the lack of $\mathcal{P}(0)$.
- We cannot account for the use of dual spaces in the definitions of some species, e.g., algebras with invariant scalar products interpreted as $V \rightarrow V^*$. (In this case, a remedy is the introduction of the cyclic operads).
- We cannot account for the structure morphisms like comultiplication $V \rightarrow V \otimes V$, and generally tensors of various co- and contravariant degrees.
- We cannot account for non-linear and not everywhere defined operations like inversion in the multiplicative group of a field.

1.4 Operads as analogs of associative algebras

In Example 1.1 and Subsection 1.1 we have shown that the associative algebras naturally form a part of the classical operads (with $\mathcal{P}(l) = 0$ for $l \geq 2$). We will now demonstrate that the total classical operad $\mathcal{P}$ is in a very definite sense an analog of associative algebra.

To do this convincingly, we must start with a definition of an associative algebra as a couple (V, m) where V is an object of the monoidal category $VECT$,

and m is an associative morphism $V \otimes V \rightarrow V$, eventually endowed with identity which is a morphism $\mathbf{1} \rightarrow V$ where $\mathbf{1}$ is the ground field considered as an identity object in $VECT$. The two categories $(VECT, \otimes)$ and ASS obtained in this way are connected by the two adjoint functors

$$\begin{aligned} \text{forget } m : ASS &\rightarrow VECT , \\ \text{free tensor algebra} : VECT &\rightarrow ASS . \end{aligned}$$

In order to present the classical linear operads in the same way we have to start with specifying an analog of the functor “forget m ”. This can be done in several ways because we can choose to forget any subset of the data given in Definition 1.1. Here we will decide that m corresponds to all γ 's. What is left then is the following category $SMOD$ of $\mathbf{S}$ -modules:

DEFINITION 1.5 *An object of $SMOD$ is a family of linear spaces $V(l), l \geq 1$, endowed with an action of $\mathbf{S}_l$.*

A morphism in $SMOD$ is a family of linear maps $V(l) \rightarrow W(l)$ compatible with the $\mathbf{S}_l$ -action.

We will sometimes say that $V(l)$ is the part of V of degree l .

LEMMA 1.1 *a) The category $SMOD$ possesses a bifunctorial product $*$ which can be defined on the objects by the following formula:*

$$V * W(n) = \bigoplus_{l=1}^n V(l) \otimes_{\mathbf{S}_l} \left(\bigoplus_{\pi: \underline{n} \rightarrow \underline{l}} \bigotimes_{i=1}^l W(|\pi^{-1}(i)|) \right) . \quad (1.13)$$

Here $\underline{n} = \{1, \dots, n\}$ and π runs over all surjective maps. The action of $\mathbf{S}_l$ must be self-explanatory, and the tensor product is taken over the group ring of $\mathbf{S}_l$.

*This product is functorially associative but not commutative so that $(SMOD, *)$ is a monoidal category. It possesses a two-sided identity object $\mathbf{1}$: the ground field placed in degree 1, zero elsewhere.*

*b) The map $V \mapsto (V, 0, 0, \dots)$ extends to a functor identifying $(VECT, \otimes, \mathbf{1})$ with a full monoidal subcategory of $(SMOD, *, \mathbf{1})$.*

Now consider an associative algebra (V, μ) , $\mu : V * V \rightarrow V$ in the monoidal category of $\mathbf{S}$ -modules. From (1.13) we see that μ is a family of maps

$$\mu(n) : \bigoplus_{l=1}^n V(l) \otimes \mathbf{S}_l \left(\bigoplus_{\pi: \underline{n} \rightarrow \underline{l}} \bigotimes_{i=1}^l V(|\pi^{-1}(i)|) \right) \rightarrow V(n), \quad n \geq 1. \quad (1.14)$$

For given $(l; k_1, \dots, k_l)$, $k_1 + \dots + k_l = n$, consider the component of (1.14) corresponding to the $\mathbf{S}_l$ -orbit of the map sending $\{1, \dots, k\}$ to $1, k_1+1, \dots, k_1+k_2$ to 2, etc. We can identify this part of the source with $V(l) \otimes V(k_1) \otimes \dots \otimes V(k_l)$ so that μ generates a family of maps

$$\gamma(k_1, \dots, k_l) : V(l) \otimes V(k_1) \otimes \dots \otimes V(k_l) \rightarrow V(k_1 + \dots + k_l). \quad (1.15)$$

PROPOSITION 1.1 *a) The associativity of μ translates into the associativity of γ 's in the sense of (1.7).*

b) The fact that μ is a morphism in $SMOD$ translates into the compatibility axioms (1.3–1.6)

c) In this way we get a functor

$$\text{Associative algebras in } (SMOD, *) \rightarrow OPER$$

which is an equivalence of categories.

There exists a similar equivalence between associative algebras with identity and unital operads.

Proof. We will now sketch a proof of the main statements in Lemma 1.1 and Proposition 1.1. In order to understand the main formula (1.13), we will show that it expresses the substitution law of “formal series in $VECT$ ”.

To be more precise, denote by $FSETS$ the category of finite non-empty sets and bijections. Let $F[\cdot] : FSETS \rightarrow VECT$ be a functor.

$FSETS$ is equivalent to its full subcategory whose objects are $\underline{n}$. Restricting F to this subcategory we get an $\mathbf{S}$ -module $V_F : V_F(n) := F[\underline{n}]$, the action of $\mathbf{S}_n$ being induced by the bijections of $\underline{n}$.

Now consider $V_F(n)$ as coefficients of the formal series defining the functor $F(\cdot) : VECT \rightarrow VECT$:

$$F(X) := \bigoplus_{n \geq 1} V_F(n) \otimes_{\mathbf{S}_n} X^{\otimes n}.$$

Such functors will be called *analytic ones*.

We will show that these constructions establish an equivalence of the three categories involved: functors $F[\cdot]$ and their morphisms, $SMOD$, analytic functors. Moreover, the composition of analytic functors is again analytic, and it induces on the coefficients exactly the $*$ -product:

$$V_{F \circ G}(n) = (V_F * V_G)(n) .$$

The equivalence of the category of functors $F[\cdot]$ and $SMOD$ is a part of general nonsense because $FSETS$ is equivalent to its subcategory of natural numbers. The only point deserving explication is the possibility to lift every $\mathbf{S}$ -module to an $F[\cdot]$ canonically without using the axiom of choice. Namely, for a finite set M with $|M| = m$ put

$$\tilde{F}[M] := F[\underline{m}] \otimes_{\mathbf{S}_m} \langle Iso(\underline{m}, M) \rangle .$$

Here $\langle Iso(\underline{m}, M) \rangle$ is the linear space freely generated by the bijections $\underline{m} \rightarrow M$. Strictly speaking, now $\tilde{F}[\underline{m}]$ is not $F[\underline{m}]$, but these $\mathbf{S}_m$ -modules are canonically isomorphic, and we forget about this subtlety and say, for example, that the $\mathbf{S}$ -module $F[\underline{m}] = X^{\otimes m}$ extends to the functor $F[M] = X^{\otimes M}$ on the category of finite sets.

The equivalence of $SMOD$ and the category of analytic functors $F(\cdot)$ also becomes a formal fact once we learn how to reconstruct functorially the coefficients $V_F(n)$. Let $F(\cdot)$ be given. Multiplication by any element λ of the ground field is an endomorphism of the identical functor of $VECT$. Hence it acts functorially on each $F(X)$, and the λ^n -eigenspace of $F(X)$ is exactly $F_n(X) := V_F(n) \otimes_{\mathbf{S}_n} X^{\otimes n}$, at least when λ is not 0 or a root of unity. Now consider the space $X_n = \langle \underline{n} \rangle$ freely generated by the vectors $e_1, \dots, e_n$. Then $e_1 \otimes \dots \otimes e_n$ generates the regular $\mathbf{S}_n$ -submodule R_n which is the image of the projector $p_n : X_n^{\otimes n} \rightarrow X_n^{\otimes n}$. Since F_n is a functor, we can define $Im(F_n(p_n)) = V_F(n) \otimes_{\mathbf{S}_n} R_n = V_F(n)$, both equalities denoting canonical isomorphisms.

We will apply this prescription to the calculation of the coefficients of the

composition of analytic functors:

$$\begin{aligned} (F \circ G)(X) &= \bigoplus_{l=1}^{\infty} V_F(X) \otimes_{S_l} \left(\bigoplus_{k=1}^{\infty} V_G(k) \otimes_{S_k} X^{\otimes k} \right)^{\otimes l} = \\ &= \bigoplus_{l=1}^{\infty} V_F(X) \otimes_{S_l} \left[\bigoplus_{k_1, \dots, k_l=1}^{\infty} (V_G(k_1) \otimes_{S_{k_1}} X^{\otimes k_1}) \otimes \dots \otimes (V_G(k_l) \otimes_{S_{k_l}} X^{\otimes k_l}) \right]. \end{aligned}$$

It follows that

$$(F \circ G)_n(X) = \bigoplus_{l=1}^{\infty} V_F(X) \otimes_{S_l} \left[\bigoplus_{k_1 + \dots + k_l = n} \bigotimes_{a=1}^l (V_G(k_a) \otimes_{S_{k_a}} X^{\otimes k_a}) \right].$$

Now we must put $X = X_n$ as above and look at the image of $(F \circ G)_n(p_n)$ or, more intuitively, at the tensor coefficients of the vectors $e_{s(1)} \otimes \dots \otimes e_{s(n)}$. Clearly, for a given l , such terms in square brackets correspond to the partitions of $\underline{n}$ into l blocks indexed by $1, \dots, l$, i.e. to the surjections $\underline{n} \rightarrow \underline{l}$ as in (1.13).

To finish the Proof, it remains to establish that the functor $F \circ G(\cdot)$ is isomorphic to the sum of $\text{Im}(F \circ G)_n(p_n)$. We leave this to the reader. $\blacksquare$

DEFINITION 1.6 *Let V be an object of $SMOD$. Put*

$$F(V) := \sum_{n=1}^{\infty} V^{*n}.$$

*There is an obvious multiplication map $V^{*m} * V^{*n} \rightarrow V^{*m+n}$ which makes $F(V)$ an associative algebra, or an operad. It is called the **free operad** generated by V (without identity).*

As in the classical linear algebra, F is adjoint to the forgetful functor $OPER \rightarrow SMOD$. This completes the analogy sketched at the beginning of Subsection 1.4.

1.5 Operads and topology: homology of moduli spaces

We will now introduce the basic operad of the quantum cohomology. Denote by $H_*(\overline{M}_{0,n+1})$ the homology space of the moduli space of stable curves of genus

zero (with coefficients in the ground field for $VECT$). We will define the classical linear operad $H_*\overline{M}_0$ by the following data:

a) $H_*\overline{M}_0(n) = H_*(\overline{M}_{0,n+1})$ for $n \geq 2$, the first component being the ground field.

In the following, it will be convenient to assume that the structure sections of $\overline{C}_{n+1} \rightarrow \overline{M}_{0,n+1}$ are labeled by $\{0, \dots, n\}$.

b) $\mathbf{S}_n$ acts upon $H_*\overline{M}_0(n)$ by renumbering the sections $x_1, \dots, x_n$.

c) The structure map

$$\begin{aligned} &\gamma(k_1, \dots, k_l) : \\ &H_*(\overline{M}_{0,l+1}) \otimes H_*(\overline{M}_{0,k_1+1}) \otimes \dots \otimes H_*(\overline{M}_{0,k_l+1}) \rightarrow H_*(\overline{M}_{0,k_1+\dots+k_l+1}) \end{aligned} \quad (1.16)$$

is induced by the embedding of the boundary stratum

$$b(k_1, \dots, k_l) : \overline{M}_{0,l+1} \times \overline{M}_{0,k_1+1} \times \dots \times \overline{M}_{0,k_l+1} \rightarrow \overline{M}_{0,k_1+\dots+k_l+1} . \quad (1.17)$$

On the level of geometric points, given $l+1$ stable labeled curves of genus zero,

$$(C; x_0, x_1, \dots, x_l); \quad (D_a; y_{0,a}, \dots, y_{k_a,a}), \quad a = 1, \dots, l .$$

$b(k_1, \dots, k_l)$ produces from them the stable curve

$$\left(C \amalg \left(\coprod_{a=1}^l D_a \right) / (\sim); \quad z_0, \dots, z_{k_1+\dots+k_l} \right) ,$$

where $(\sim)$ is the equivalence relation gluing x_a and $y_{0,a}$ for all $a = 1, \dots, l$, and furthermore

$$z_0 = x_0, (z_1, \dots, z_{k_1+\dots+k_l}) = (y_{1,1}, \dots, y_{k_1,1}; \dots; y_{1,a}, \dots, y_{k_a,a}) .$$

Operadic axioms for $H_*\overline{M}_0$ follow from their evident versions for the spaces $\overline{M}_{0n}$.

REMARK 1.3 *What we are actually saying here is that we can define the more general notion of operad by replacing the basic category $VECT$ by any symmetric monoidal category, eventually with the identity object, and that the moduli spaces form such an operad. The homology functor (with respect to the pushforward maps)*

from the monoidal category of manifolds to $(VECT, \otimes)$ then produces from a geometric operad the classical linear operad. This viewpoint will be discussed in more detail in the following Section.

Notice in conclusion that $H_*\overline{M}_0$ is endowed with important additional structures. Namely, the components of this operad are in fact coalgebras (pushforward with respect to the diagonal map), and compositions (1.16) as well as representations of $\mathbf{S}_n$ are coalgebra morphisms. This is the intrinsic reason for the existence of the operation of the tensor product on the category of $H_*\overline{M}_0$ -algebras.

2 Operads and Trees

In this Section we sketch in their natural generality several themes which have already emerged in the previous Section. Briefly speaking, there are many useful types of operads, and each type is determined by the choice of two categories:

- 1) Basic symmetric monoidal category $(\mathcal{C}, \boxtimes)$ replacing $(VECT, \otimes)$ which supports the classical linear operads.
- 2) A category of (labeled) graphs Γ reflecting the combinatorics of the operadic data and axioms.

A concrete operad from this viewpoint is a functor $\Gamma \rightarrow \mathcal{C}$.

To clarify the role of $\mathcal{C}$, we first explain how to extend Definition 1.1.

2.1 May's operads in a monoidal category

Let us recall (see [5]) that a *symmetric monoidal category* $(\mathcal{C}, \boxtimes)$ is a category endowed with the bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ together with an involutive commutativity constraint and an associativity constraint. Taken together, they define a family of compatible and functorial isomorphisms $s_* : X_1 \boxtimes \cdots \boxtimes X_n \xrightarrow{\sim} X_{s^{-1}(1)} \boxtimes \cdots \boxtimes X_{s^{-1}(n)}$, for any objects $X_1, \dots, X_n$ of $\mathcal{C}$ and all $s \in \mathbf{S}_n$.

Most of our monoidal categories will have an identity object $\mathbf{1}_{\mathcal{C}} = \mathbf{1}$. The functors $\mathbf{1} \boxtimes$ and $\boxtimes \mathbf{1} : \mathcal{C} \rightarrow \mathcal{C}$ are canonically isomorphic to the identity functor.

In order to be able to extend the constructions of Subsection 1.4, we will assume that $\mathcal{C}$ has small colimits preserved by any functor $X \boxtimes$. In particular, $\mathcal{C}$ must have an initial object 0.

We can now define a *classical operad* $\mathcal{P}$ in $\mathcal{C}$ by closely following Definition 1.1. Components $\mathcal{P}(n)$ will be objects of $\mathcal{C}$ endowed with the action of $\mathbf{S}_n$, $\otimes$ will be replaced by $\boxtimes$, and operadic multiplications γ will be morphisms in $\mathcal{C}$. Axioms A) – C) must be written down as commutative diagrams, involving in particular permutation isomorphisms of tensor products in $\mathcal{C}$.

A neater version of the definition is again obtained by passing to the category $\mathcal{SC}$ every object which is a family of $\mathbf{S}_n$ -objects $\mathcal{P}(n)$ in $\mathcal{C}$ given for $n \geq 1$. To be able to write it as a sum of its components, we will require that $\mathcal{C}$ has small limits. The category $\mathcal{SC}$ admits a *non-symmetric* monoidal structure $*$, furnished by the formula (1.13). It has the unit object $\mathbf{1}_{\mathcal{SC}}$ with $\mathbf{1}$ as the first component, 0 elsewhere. An associative monoid in $\mathcal{SC}$ is a pair $(\mathcal{P}, \mu)$ where $\mu : \mathcal{P} * \mathcal{P} \rightarrow \mathcal{P}$ is an associative multiplication. Giving an additional morphism $\mathbf{1} \rightarrow \mathcal{P}$ with the usual properties defines unital monoids. An analog of Proposition 1.1 holds true, establishing the equivalence of the category of associative monoids in $(\mathcal{SC}, *)$ and the category of classical operads in $\mathcal{C}$. However, the proof of Proposition 1.1 must be changed, because we have used in it not only the monoidal structure of $\mathbf{VECT}$ but the linear structure and the language of elements as well. This can be avoided in different ways. Here we will take this fact for granted, and we leave to the reader the transposition of other constructions of Section 1 to the present context.

REMARK 2.1 *Let us consider the main classes of monoidal categories. Sets with direct product and linear spaces with tensor product form two archetypal classes of symmetric monoidal categories.*

Variations include imposing additional structure on the objects. Sets more often appear endowed with a topology or manifold structure (in smooth or analytic category). Linear spaces come equipped with grading and/or differential. In this way we get classical topological operads, classical operads in the category of complexes, and so on. Monoidal functors between symmetric monoidal categories extend to the respective categories of operads.

2.2 Oriented trees as substitution schemes

Let T be a *tree* with at least two flags at each vertex. Orient T by choosing one tail as *root* and declaring that direction to the root is positive. Then every vertex has at least one incoming flag and exactly one outgoing flag. Label each

vertex of T by a symbol of the function whose arguments are labeled by the incoming flags of this vertex, and whose value labels the outgoing flag f and also its l -image, that is, the other half of the edge if f belongs to an edge. Then the whole tree symbolizes a computation, or substitution scheme. The input values are assigned to the incoming tails of T , and the output value is assigned to the root. For example, one vertex tree with n incoming tails symbolizes $f(x_1, \dots, x_n)$ and the $(m+1)$ -vertex tree with the appropriate distribution of flags symbolizes $f(g_1(x_1^{(1)}, \dots, x_{n_1}^{(1)}), \dots, g_m(x_1^{(m)}, \dots, x_{n_m}^{(m)}))$.

If we label flags by objects of a symmetric monoidal category and label each vertex v by a morphism mapping the $\boxtimes$ -product of the labels of incoming flags to the label of the outgoing flag, the tree will describe the respective composite morphism from the $\boxtimes$ -product of input objects to the output object.

If $\boxtimes$ is not supposed to be symmetric, we must assume that all sets of incoming flags of each vertex are totally ordered. For a symmetric $\boxtimes$ -product, the respective actions of symmetric groups on arguments of various levels can be succinctly described by saying that this construction is functorial on the category of oriented trees with isomorphisms compatible with orientation.

2.3 Trees and $\ast$ -product

Let $(\mathcal{C}, \boxtimes)$ be a *symmetric monoidal category*, V an object of *non-symmetric monoidal category* $(\mathcal{SC}, \ast)$, and $F(V) = \coprod_{n=1}^{\infty} V^{\ast n}$ the free operad generated by V , as in Definition 1.6. We can define $\boxtimes$ -products indexed by arbitrary finite sets and extend $n \mapsto V(n)$ to a functor $T \mapsto V(T)$ on the category of non-empty finite sets and their bijections. For an oriented tree as above put

$$V(T) := \boxtimes_{v \in V_T} V(F_T^{\text{in}}(v)) . \quad (2.1)$$

Then we have functorial isomorphisms

$$F(V)(n) = \coprod_{\{n\text{-trees}\}T/(iso)} V(T) . \quad (2.2)$$

Here n -trees are oriented trees with the set $\{1, \dots, n\}$ of incoming tails.

This statement summarizes in a more conceptual way the bookkeeping scheme described above. It can be deduced with some pain from the formalism of analytic

functors as in Subsection 1.4. We will also reproduce the relevant combinatorics in the context of formal series below, in Section 12 dedicated to sums over trees.

2.4 Combinatorial trees

In Subsections 2.2–2.3 trees were defined in a purely abstract way: T is the free plain operad on the terminal object of $\mathbf{Set}_f^{\mathbb{N}}$, and an n -leafed tree is an element of T_n . But we give here a graph-theoretic definition of (finite, rooted, planar) tree.

The main subtlety is that the trees we use are not quite finite graphs in the usual sense: some of the edges have a vertex at only one of their ends. This suggests the following definitions.

DEFINITION 2.1

A (planar) **input – output graph** (Fig. 1(a)) consists of

- a finite set V (the **vertices**)
- a finite set E (the **edges**), a subset $I \subseteq E$ (the **input edges**), and an element $o \in E$ (the **output edge**)
- a function $s : E \setminus I \rightarrow V$ (**source**) and a function $t : E \setminus \{o\} \rightarrow V$ (**target**)
- for each $v \in V$, a total order $\leq$ on $t^{-1}\{v\}$.

We write $v \xrightarrow{e}$ to mean that e is a non-input edge with $s(e) = v$, and similarly $\xrightarrow{e} v'$ to mean that e is a non-output edge with $t(e) = v'$, and of course $v \xrightarrow{e} v'$ to mean that e is a non-input, non-output edge with $s(e) = v$ and $t(e) = v'$.

A tree is roughly speaking a connected, simply connected graph, and the following notion of path allows us to express this.

DEFINITION 2.2 A **path** from a vertex v to an edge e in an input-output graph is a diagram

$$v = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \dots \xrightarrow{e_{l-1}} v_l \xrightarrow{e_l=e}$$

in the graph. That is, a path from v to e consists of

- an integer $l \geq 1$

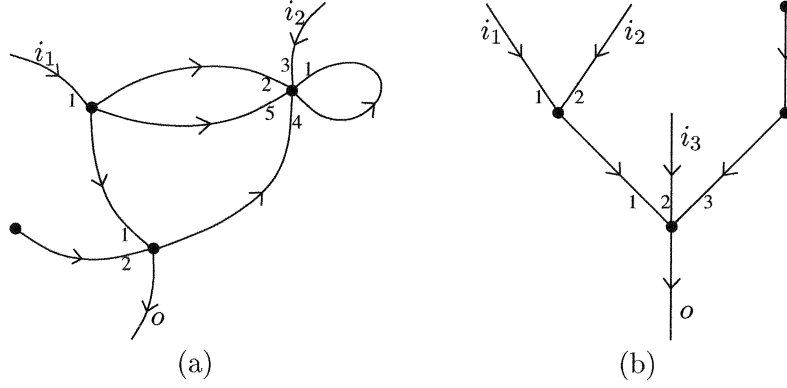


Figure 1: (a) Input-output graph with 4 vertices and 2 input edges i_1, i_2 , (b) combinatorial tree with 4 vertices and 3 input edges i_1, i_2, i_3 . In both, the numbers indicate the order on the edges arriving at each vertex.

- a sequence $(v_1, v_2, \dots, v_l)$ of vertices with $v_1 = v$
- a sequence $(e_1, \dots, e_{l-1}, e_l)$ of edges with $e_l = e$

such that

$$v_1 = s(e_1), t(e_1) = v_2 = s(e_2), \dots, t(e_{l-1}) = v_l = s(e_l)$$

and all of these sources and targets are defined.

DEFINITION 2.3 A **combinatorial tree** is an input-output graph such that for every vertex v , there is precisely one path from v to the output edge.

Fig. 1(b) shows a combinatorial tree. The ordering of the edges arriving at each vertex encodes the planar embedding. ‘Tree’ is an abbreviation for ‘finite, rooted, planar tree’. If we were doing symmetric operads then we would use non-planar trees, if we were doing cyclic operads then we would use non-rooted trees, and so on.

2.5 Geometric interpretation of trees

Let T_n be the set of rooted, planar binary trees with n interior nodes (and thus $n + 1$ leaves). The *Tamari order* (see [2]) on T_n is the partial order whose cover

relations are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus



is an increasing chain in T_3 (the moving vertices are marked with dots). Only basic properties of the Tamari order are needed in this Subsection; their proofs will be provided. For more properties, see Chapters 3–7. Figure 2 shows the Tamari order on T_3 and T_4 .

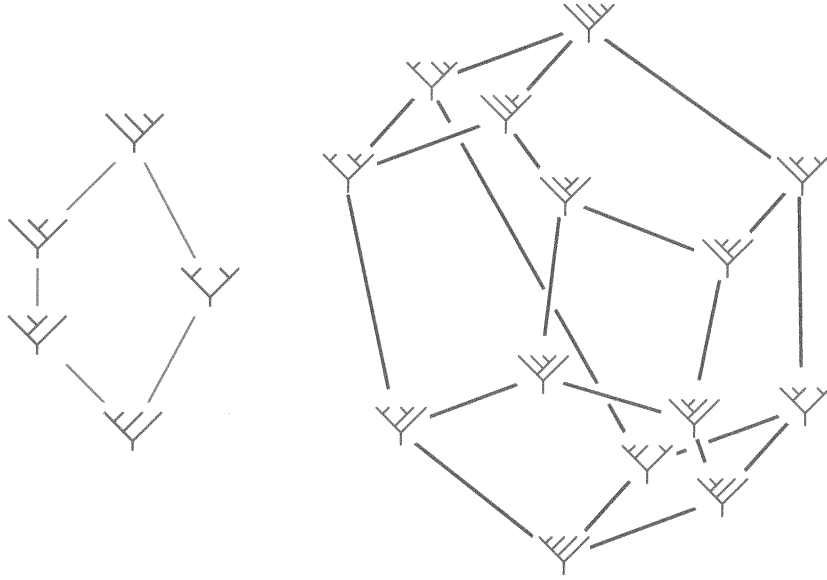


Figure 2: The Tamari order on T_3 and T_4 .

Let 1_n be the minimum tree in T_n . It is called a *right comb* as all of its leaves are right pointing:

$$1_4 = \begin{array}{c} \diagup \diagup \diagup \\ | \\ \diagup \end{array}, \quad 1_7 = \begin{array}{c} \diagup \diagup \diagup \diagup \diagup \diagup \diagup \\ | \\ \diagup \end{array}.$$

Given trees $s \in T_p$ and $t \in T_q$, the tree $s \vee t \in T_{p+q+1}$ is obtained by grafting the root of s onto the left leaf of the tree $\vee$ and the root of t onto its right leaf. Below

we display trees s , t , and $s \vee t$, indicating the position of the grafts with dots.



For $n > 0$, every tree $t \in T_n$ has a unique decomposition $t = t_l \vee t_r$ with $t_l \in T_p$, $t_r \in T_q$, and $n = p + q + 1$. Thus T_n is in bijection with $\bigsqcup_{p+q=n-1} T_p \times T_q$, and since $T_0 = \{|\}$ and $T_1 = \{\vee\}$, we shall see in Section 3.5 that T_n contains the Catalan number $\frac{(2n)!}{n!(n+1)!}$ of trees.

The Hasse diagram of T_n is isomorphic to the 1-skeleton of the *associahedron* $\mathcal{A}_n$, an $(n-1)$ -dimensional polytope. (See [6] and [7].) The faces of $\mathcal{A}_n$ are in one-to-one correspondence with collections of non-intersecting diagonals of a polygon with $n+2$ sides (an $(n+2)$ -gon). Equivalently, the faces of $\mathcal{A}_n$ correspond to polygonal subdivisions of an $n+2$ -gon with facets corresponding to diagonals and vertices to triangulations. The dual graph of a polygonal subdivision is a planar tree and the dual graph of a triangulation is a planar binary tree. If we distinguish one edge to be the root edge, the trees are rooted, and this furnishes a bijection between the vertices of $\mathcal{A}_n$ and T_n . Figure 3 shows two views of the associahedron $\mathcal{A}_3$, the first as polygonal subdivisions of the pentagon, and the second as the corresponding dual graphs (planar trees). The root is at the bottom.

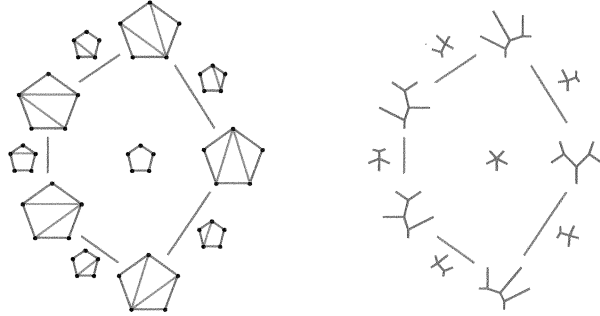


Figure 3: Two views of the associahedron $\mathcal{A}_3$

Let $\mathfrak{S}_n$ be the group of permutations of $[n]$ which denotes the set $\{1, 2, \dots, n\}$. We describe the map $\lambda: \mathfrak{S}_n \rightarrow T_n$ in terms of triangulations of the $(n+2)$ -gon where we label the vertices with $0, 1, \dots, n, n+1$ beginning with the left vertex of

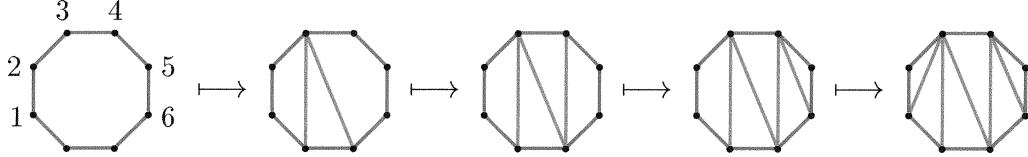
the root edge and proceeding clockwise. Let $\sigma \in \mathfrak{S}_n$ and set $w_i := \sigma^{-1}(n+1-i)$, for $i = 1, \dots, n$. This records the positions of the values of σ taken in decreasing order. We inductively construct the triangulation, beginning with the empty triangulation consisting of the root edge, and after i steps we have a triangulation τ_i of the polygon

$$P_i := \text{Conv}\{0, n+1, w_1, \dots, w_i\}.$$

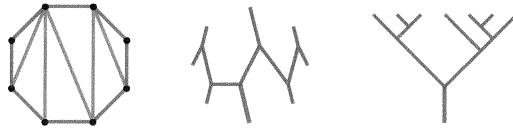
Some edges of P_i will be edges of the original $(n+2)$ -gon and others will be diagonals. Each diagonal cuts the $(n+2)$ -gon into two pieces, one containing P_i and the other a polygon which is not yet triangulated and whose root edge we take to be that diagonal. Subsequent steps add to the triangulation τ_i and its support P_i .

First set $\tau_1 := \text{Conv}\{0, n+1, w_1\}$, the triangle with base the root edge and apex the vertex $w_1 = \sigma^{-1}(n)$. Set $P_1 := \tau_1$ and continue. After i steps we have constructed τ_i and P_i in such a way that the vertex w_{i+1} is not in P_i . Hence it must lie in some untriangulated polygon consisting of some consecutive edges of the $(n+2)$ -gon and a diagonal that is an edge of P_i . Add the join of the vertex w_{i+1} and the diagonal to the triangulation to obtain a triangulation τ_{i+1} of the polygon P_{i+1} . The process terminates when $i = n$.

For example, we display this process for the permutation $\sigma = 316524$, where we label the vertices of the first octagon:

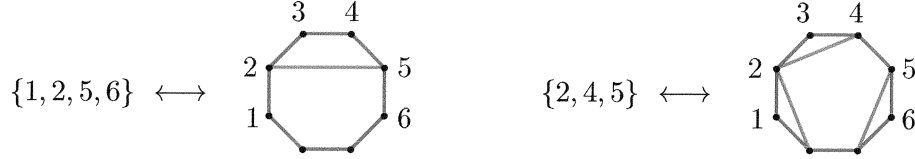


The last two steps are suppressed as they add no new diagonals. The dual graph to the triangulation τ_n is the planar binary tree $\lambda(\sigma)$. Here is the triangulation, its dual graph, and a ‘straightened’ version, which we recognize as the tree $\lambda(316524)$.



A subset S of $[n]$ determines a face Φ_S of the associahedron $\mathcal{A}_n$ as follows. Suppose that we label the vertices of the $(n+2)$ -gon as above. Then the vertices

labeled $0, n+1$ and those labeled by S form a $(\#S + 2)$ -gon whose edges include a set E of non-crossing diagonals of the original $(n+2)$ -gon. These diagonals determine the face Φ_S of $\mathcal{A}_n$ corresponding to S . We give two examples of this association when $n = 6$ below.



We determine the image of f_ζ using the above description of the map $\lambda: \mathfrak{S}_n \rightarrow T_n$. We say that a face of $\mathcal{A}_{p+q}$ of the form Φ_S with $\#S = q$ has *type* (p, q) . If a face has a type, this type is unique. A permutation $\zeta \in \mathfrak{S}^{(p,q)}$ is uniquely determined by the set $\zeta\{p+1, \dots, p+q\}$. Therefore, a face of type (p, q) is the image of f_ζ for a unique permutation $\zeta \in \mathfrak{S}^{(p,q)}$. This allows us to speak of the vertex of the face corresponding to a pair $(s, t) \in T_p \times T_q$ (under f_ζ).

2.6 Classical operads as functors

Denote by **Tree**_{clas} as the category whose objects are finite rooted trees with the following properties: a) the multiplicity of each vertex is at least two; b) at each vertex either all incoming flags are halves of edges, or all incoming flags are tails. Morphisms are generated by the following two classes of maps:

- a) Isomorphisms compatible with orientation.
- b) Contraction of all edges having a common vertex with some outgoing flag and keeping orientation.

More formally, a morphism $\varphi: \sigma \rightarrow \tau$ consists of two maps $\varphi_V: V_\sigma \rightarrow V_\tau$ and $\varphi^F: F_\tau \rightarrow F_\sigma$ compatible with boundaries and involutions and such that φ^F sends tails to tails. Composition of the morphisms corresponds to the composition of the induced maps on vertices and flags. A morphism contracts an edge e if φ_V glues its vertices, and both flags of this edge do not belong to the image of φ^F .

Contractions of different edges commute in an evident sense.

Let v be a vertex of a rooted tree T . Its *star* T_v is a one-vertex tree with vertex v , tails $F_T(v)$, and the outgoing flag as a root.

PROPOSITION 2.1 *The category of classical linear operads (without identity) in a symmetric monoidal category $(\mathcal{C}, \boxtimes)$ is equivalent to the category of functors $\mathcal{P} : \mathbf{Tree}_{\text{class}} \rightarrow \mathcal{C}$ isomorphic to a functor satisfying the following condition:*

$$\mathcal{P}(T) = \boxtimes_{v \in V_T} \mathcal{P}(T_v) . \quad (2.3)$$

Sketch of Proof. a) *From functors to operads.* Given such a functor $\mathcal{P}$, we construct the data of Definition 1.1 in the following way: $\mathcal{P}(l) := \mathcal{P}(T_l)$ where T_l is the one-vertex tree with tails $0, 1, 2, \dots, l$ and root 0 . The action of $\mathbf{S}_l$ corresponds to the automorphisms of T_l permuting the tails $1, \dots, l$. The multiplication map $\gamma(k_1, \dots, k_l)$ corresponds to the morphism contracting all edges $\sigma \rightarrow \tau_{k_1 + \dots + k_l}$, where σ has $l + 1$ vertices and l edges and the tails are distributed in an obvious way. The relations A) and B) follow from the functoriality.

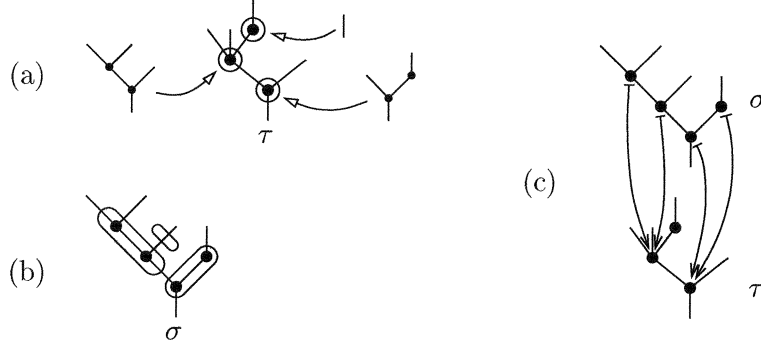
b) *From operads to functors.* Given an operad $(\mathcal{P}(n), \gamma)$, we first extend it to the functor from finite sets to $\mathcal{C}$, then define $\mathcal{P}(T)$ by (2.1), and finally use γ in order to define $\mathcal{P}$ on morphisms contracting all edges having a common vertex with some outgoing flag. ■

DEFINITION 2.4 *From the graph-theoretic viewpoint it would be more natural to allow all rooted trees with $|v| \geq 2$ as objects, and contractions of any subset of edges as morphisms. The functors from this category $\mathbf{Tree}_M$ to $\mathcal{C}$ satisfying (2.3) (up to functor isomorphism) are called **Markl's operads**.*

REMARK 2.2 *Consider now the category $\mathbf{Tree}_{\text{cyc}}$ of finite non-rooted trees with $|v| \geq 2$, with morphisms generated by contraction of edges and isomorphisms. Neither root nor orientation is a part of the structure. Functors $\mathbf{Tree}_{\text{cyc}} \rightarrow \mathcal{C}$ satisfying (2.3) are essentially cyclic operads in the sense of [9]. The most essential new feature of cyclic operads is the action of $\mathbf{S}_{l+1}$ upon $\mathcal{P}(l)$.*

2.7 Classifying space of the category of stable trees

Let us consider a graphical definition of a category of trees. By Definition 2.3, $\mathbf{tr}$ is the free plain operad on the terminal object of $\mathbf{Set}_f^{\mathbb{N}}$, and an n -leafed tree is an element of $\mathbf{tr}_n$. As we saw, the sets $\mathbf{tr}_n$ also admit the following recursive description:

Figure 4: Three pictures of a map in $\mathbf{Tree}_4$

- $| \in \mathbf{tr}_1$
- if $n, k_1, \dots, k_n \in \mathbb{N}$ and $\tau_1 \in \mathbf{tr}_{k_1}, \dots, \tau_n \in \mathbf{tr}_{k_n}$ then $(\tau_1, \dots, \tau_n) \in \mathbf{tr}_{k_1 + \dots + k_n}$.

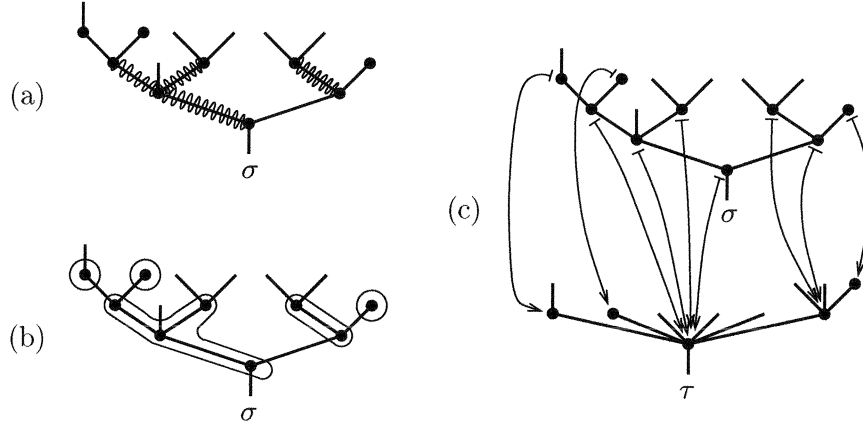
A category of trees $\mathbf{Tree}$ is the disjoint union $\coprod_{n \in \mathbb{N}} \mathbf{Tree}_n$. An object of $\mathbf{Tree}_n$ is an n -leafed tree. The set of maps in $\mathbf{Tree}_n$ is

$$(T_2^2 1)(n) = (T_2(\mathbf{tr}))(n),$$

that is, a map is an n -leafed tree τ in which each k -ary vertex v has assigned to it a k -leafed tree σ_v ; the domain of the map is the tree obtained by gluing the σ_v 's together in the way dictated by the shape of τ , and the codomain is τ itself. Put another way, what a map does is to take a tree σ (the domain), partition it into subtrees σ_v , and in (c) as something looking more like a function.



We will return to the third point of view later; for now, just observe that there is an induced function from the vertices of σ to the vertices of τ , in which the inverse image of a vertex v of τ is the set of vertices of σ_v . In some texts a map

Figure 5: Three pictures of an epic in $\mathbf{Tree}_6$

of trees is described as something that ‘contracts some internal edges’. (Here an *internal edge* is an edge that is not the root or a leaf; maps of trees keep the root and leaves fixed. To ‘contract’ an internal edge means to shrink it down to a vertex.) With one important caveat, this is what our maps of trees do: for in a map $\sigma \rightarrow \tau$, the replacement of each partitioning subtree σ_v by the corolla with the same number of leaves amounts to the contraction of all the internal edges of σ_v . For example, Fig. 5(a) shows a tree σ with some of its edges marked for contraction, and Figs. 5(b) and 5(c) show the corresponding maps $\sigma \rightarrow \tau$ in two different styles (as in Figs. 4(b) and (c)); so τ is the tree obtained by contracting the marked edges of σ . The caveat is that some of the σ_v ’s may be the trivial tree, and these are replaced by the 1-leaved corolla $\bullet$. This does *not* amount to the contraction of internal edges: it is, rather, the addition of a vertex to the middle of a (possibly external) edge. Any map of trees can be viewed as a combination of contractions of internal edges and additions of vertices to existing edges. For example, the map illustrated in Fig. 4 contracts two internal edges and adds a vertex to one edge.

Some further understanding of the category of trees can be gained by considering just those trees in which each vertex has at least two branches coming up out of it. We shall call these ‘stable trees’, following Kontsevich and Manin [10]. Formally, $\mathbf{StTree}_n$ is the full subcategory of $\mathbf{Tree}_n$ with objects defined by the



Figure 6: (a) The category of 3-leafed stable trees, and (b) its classifying space

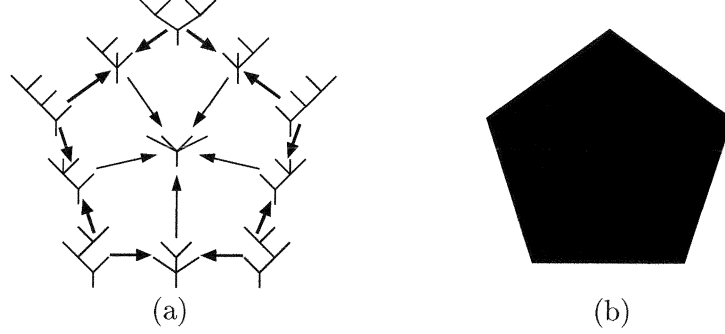


Figure 7: (a) The category of 4-leafed stable trees, and (b) its classifying space

recursive clauses

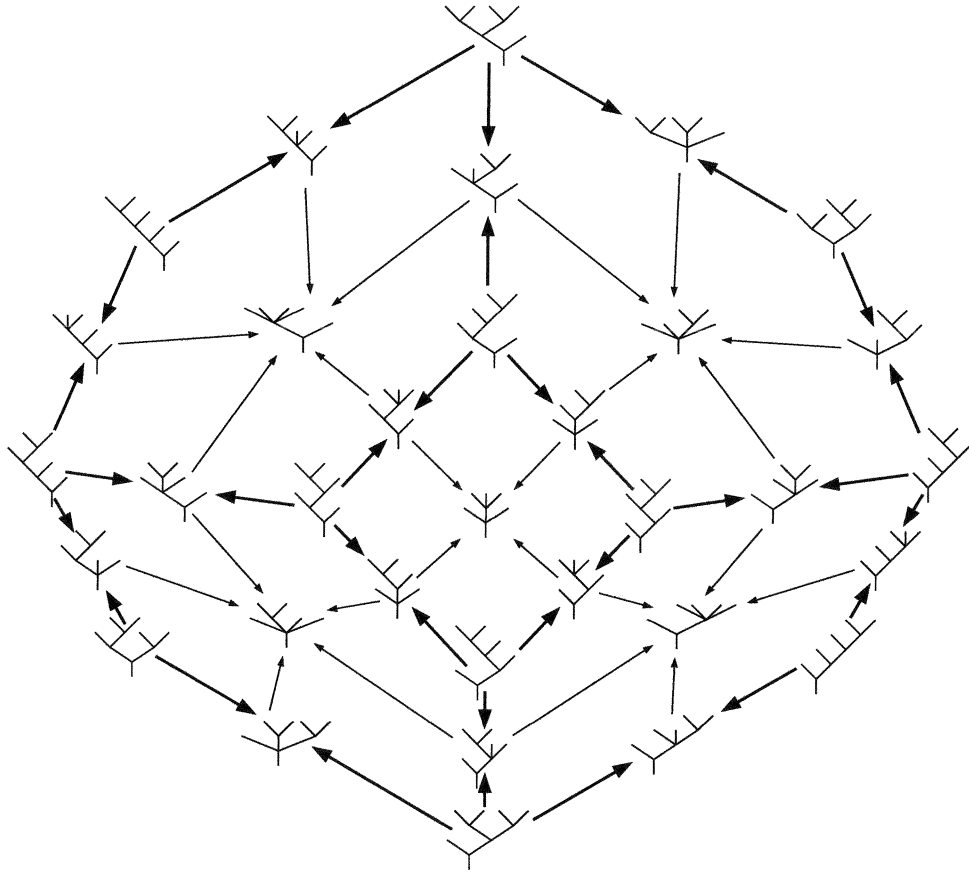
- $| \in \mathbf{StTree}_1$
- if $n \geq 2$, $k_1, \dots, k_n \in \mathbb{N}$, and $T_1 \in \mathbf{StTree}_{k_1}, \dots, T_n \in \mathbf{StTree}_{k_n}$ then $(T_1, \dots, T_n) \in \mathbf{StTree}_{k_1 + \dots + k_n}$,

and an n -leafed stable tree is an object of $\mathbf{StTree}_n$. Since a stable tree can contain no subtree of the form $\begin{array}{c} | \\ \bullet \end{array}$, all maps between stable trees are ‘surjections’, that is, consist of just contractions of internal edges, without insertions of new vertices.

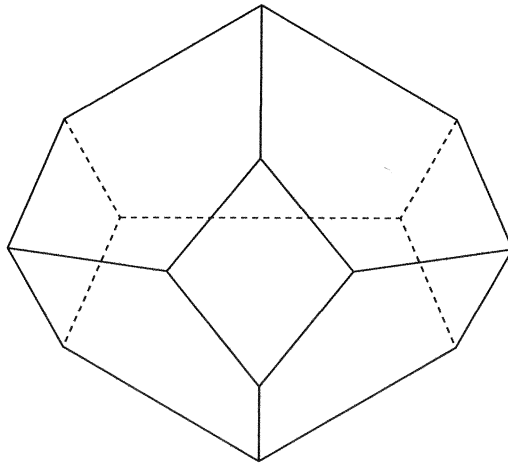
The first few categories $\mathbf{StTree}_n$ are trivial:

$$\begin{aligned} \mathbf{StTree}_0 &= \emptyset, \\ \mathbf{StTree}_1 &= \{ | \}, \\ \mathbf{StTree}_2 &= \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right\} \end{aligned}$$

where in each case there are no arrows except for identities. The cases $n = 3, 4$, and 5 are illustrated in Figs. 6(a), 7(a), and 8(a). Identity arrows are not shown, and the categories $\mathbf{StTree}_n$ are ordered sets: all diagrams commute. Vertices are



(a)



(b)

Figure 8: (a) About half of the category of 5-leafed stable trees, and (b) the classifying space of the whole category

also omitted; since the trees are stable, this does not cause ambiguity. Parts (b) of the figures show the classifying spaces of these categories, solid polytopes of dimensions 1, 2 and 3. In the case of 5-leafed trees (Fig. 8) only about half of the category is shown, corresponding to the front faces of the polytope; the back faces and the terminal object of the category (the 5-leafed corolla), which sits at the centre of the polytope, are hidden. The whole polytope has 6 pentagonal faces, 3 square faces, and 3-fold rotational symmetry about the central vertical axis.

For $n \leq 5$, the classifying space $B(\mathbf{StTree}_n)$ is homeomorphic to the associahedron $\mathcal{A}_n$ (see [11] and Figure 3 above), and it seems very likely that this persists for all $n \in \mathbb{N}$. Indeed, the family of categories $(\mathbf{StTree}_n)_{n \in \mathbb{N}}$ forms a sub-**Cat**-operad **STTR** of **Cat**-operad **TR**, and the classifying space functor $B : \mathbf{Cat} \rightarrow \mathbf{Top}$ preserves finite products, so there is a (non-symmetric) topological operad $B(\mathbf{STTR})$ whose n th part is the classifying space of $\mathbf{StTree}_n$. (To make B preserve finite products we must interpret **Top** as the category of compactly generated or Kelley spaces: see [12] and [13].) This operad $B(\mathbf{STTR})$ is presumably isomorphic to Stasheff's operad $K = (K_n)_{n \in \mathbb{N}}$. A K -algebra is called an A_∞ -space, and should be thought of as an up-to-homotopy version of a topological semigroup; the basic example is a loop space.

The categories $\mathbf{StTree}_n$ also give rise to the notion of an A_∞ -algebra (see [11]). For each $n \in \mathbb{N}$, there is a chain complex $P(n)$ whose degree k part is the free abelian group on the set of n -leafed stable trees with $(n - k - 1)$ vertices.

When the signs are chosen appropriately this defines an operad P of chain complexes. A P -algebra is called an A_∞ -algebra, to be thought of as an up-to-homotopy differential graded non-unital algebra; the usual example is the singular chain complex of an A_∞ -space. A P -category is called an A_∞ -category (see [14]), and consists of a collection of objects, a chain complex $\mathrm{Hom}(a, b)$ for each pair (a, b) of objects, maps defining binary composition, chain homotopies witnessing that this composition is associative up to homotopy, further homotopies witnessing that the previous homotopies obey the pentagon law up to homotopy, and so on.

Finally, since the polytopes $K_n = B(\mathbf{StTree}_n)$ describe higher associativity conditions, they also arise in definitions of higher-dimensional category. For example, the pentagon K_4 occurs in the classical definition of bicategory [14], and the polyhedron K_5 occurs as the 'non-abelian 4-cocycle condition' in Gordon, Power and Street's definition of tricategory [15].

We have already described operad of trees as the set $\mathbf{tr}_n$ of n -leafed trees. Maps $\sigma \longrightarrow \tau$ between trees are described by induction on the structure of τ :

- if $\tau = |$ then there is only one map into τ ; it has domain $|$ and we write it as $1_| : | \longrightarrow |$
- if $\tau = (\tau_1, \dots, \tau_n)$ for $\tau_1 \in \mathbf{tr}_{k_1}, \dots, \tau_n \in \mathbf{tr}_{k_n}$ then a map $\sigma \longrightarrow \tau$ consists of trees $\rho \in \mathbf{tr}_n, \rho_1 \in \mathbf{tr}_{k_1}, \dots, \rho_n \in \mathbf{tr}_{k_n}$ such that $\sigma = \rho^\circ(\rho_1, \dots, \rho_n)$, together with maps

$$\rho_1 \xrightarrow{\theta_1} \tau_1, \quad \dots, \quad \rho_n \xrightarrow{\theta_n} \tau_n,$$

and we write this map as

$$\sigma = \rho^\circ(\rho_1, \dots, \rho_n) \xrightarrow{!_{\rho^*(\theta_1, \dots, \theta_n)}} (\tau_1, \dots, \tau_n) = \tau. \quad (2.4)$$

It follows easily that the n -leafed corolla $\nu_n = (|, \dots, |)$ is the terminal object of $\mathbf{Tree}_n$: the unique map from $\sigma \in \mathbf{tr}_n$ to ν_n is written as $!_{\sigma} * (1_|, \dots, 1_|)$. The rest of the structure of the **Cat**-operad $\mathbf{TR}$ can be described in a similarly explicit recursive fashion.

To make precise the intuition that a map of trees is a function of some sort, functors

$$V : \mathbf{Tree} \longrightarrow \mathbf{Set}_f, \quad E : \mathbf{Tree}^{\text{op}} \longrightarrow \mathbf{Set}_f$$

can be defined, encoding what happens on vertices and edges respectively. Both functors turn out to be faithful, which means that a map of trees is completely determined by its effect on either vertices or edges. The following account of V and E is just a sketch.

The more obvious of the two is the vertex functor V , defined on objects by

- $V(|) = \emptyset$
- $V((\tau_1, \dots, \tau_n)) = 1 + V(\tau_1) + \dots + V(\tau_n)$.

The edge functor E can be defined by first defining a functor $E_n : \mathbf{Tree}(n)^{\text{op}} \longrightarrow (n+1)/\mathbf{Set}_f$ for each $n \in \mathbb{N}$, where $(n+1)/\mathbf{Set}_f$ is the category of sets equipped with $(n+1)$ ordered marked points. This definition is again by induction, the idea being that E_n associates to a tree its edge-set with the n input edges and

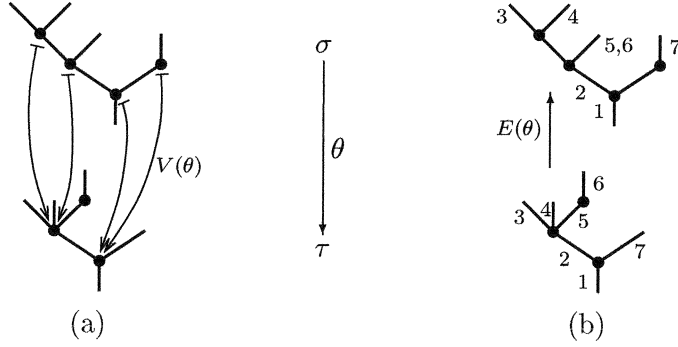


Figure 9: The effect on (a) vertices and (b) edges of a certain map of 4-leafed trees

the one output edge (root) distinguished. Fig. 9 illustrates a map $\theta : \sigma \rightarrow \tau$ in $\mathbf{Tree}(4)$; part (a) (= Fig. 4(c)) shows its effect $V(\theta)$ on vertices; part (b) shows $E(\theta)$, taking $E(\tau) = \{1, \dots, 7\}$ and labelling the image of $i \in \{1, \dots, 7\}$ under $E(\theta)$ by an i on the edge $(E(\theta))(i)$ of σ .

A map of trees will be called *surjective* if it is built up from contractions of internal edges. Formally, the *surjective* maps in $\mathbf{Tree}$ are defined by:

- $1_{|} : | \rightarrow |$ is surjective
- with notation as in (2.4), $!_{\rho} * (\theta_1, \dots, \theta_n)$ is surjective if and only if each θ_i is surjective and $\rho \neq |$.

The crucial part is the last: the unique map $!_{\rho}$ from $\rho \in \mathbf{tr}_n$ to the corolla ν_n is made up of edge-contractions just as long as ρ is not the unit tree $|$.

Dually, a map of trees is *injective* if, informally, it is built up from adding vertices to the middle of edges. Formally,

- $1_{|} : | \rightarrow |$ is injective
- with notation as above, $!_{\rho} * (\theta_1, \dots, \theta_n)$ is injective if and only if each θ_i is injective and ρ is either ν_n or $|$ (the latter only being possible if $n = 1$).

3 Unital Operads

Although operads and the most of related structures were defined in Sections 1 and 2 as an arbitrary symmetric monoidal category with countable coproducts, in Sections 3–11 we decided to follow the choice of [16] and formulate precise definitions only for the category $\mathbf{Mod}_{\mathbf{k}} = (\mathbf{Mod}_{\mathbf{k}}, \otimes)$ of modules over a commutative unital ring $\mathbf{k}$, with the monoidal structure given by the tensor product $\otimes = \otimes_{\mathbf{k}}$ over $\mathbf{k}$. The reason for such a decision was to give, in Section 6, a clean construction of free operads. In a general monoidal category, this construction involves the unordered $\odot$ -product [17] so the free operad is then a double colimit, see [17]. Our choice also allows us to write formulas involving maps in terms of elements, which is sometimes a welcome simplification. We believe that the reader can easily reformulate next definitions and notations into other monoidal categories used in Sections 1–2 and 12–13 (see, also [17, 18]).

Let $\mathbf{k}[\Sigma_n]$ denote the $\mathbf{k}$ -group ring of the symmetric group Σ_n .

DEFINITION 3.1 (MAY’S OPERAD) *An operad in the category of $\mathbf{k}$ -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of right $\mathbf{k}[\Sigma_n]$ -modules, together with $\mathbf{k}$ -linear maps (operadic compositions)*

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \rightarrow \mathcal{P}(k_1 + \cdots + k_n), \quad (3.1)$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, and a unit map $\eta : \mathbf{k} \rightarrow \mathcal{P}(1)$. These data fulfill the following axioms.

Associativity. Let $n \geq 1$ and let $m_1, \dots, m_n$ and $k_1, \dots, k_m$, where $m := m_1 + \cdots + m_n$, be non-negative integers. Then the following diagram, in which $g_s := m_1 + \cdots + m_{s-1}$ and $h_s = k_{g_s+1} \cdots + k_{g_s+1}$, for $1 \leq s \leq n$, commutes.

$$\begin{array}{ccc}
\left(\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(m_s) \right) \otimes \bigotimes_{r=1}^m \mathcal{P}(k_r) & \xrightarrow{\gamma \otimes id} & \mathcal{P}(m) \otimes \bigotimes_{r=1}^m \mathcal{P}(k_r) \\
\downarrow \text{shuffle} & & \downarrow \gamma \\
& & \mathcal{P}(k_1 + \cdots + k_m) \\
& & \uparrow \gamma \\
\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \left(\mathcal{P}(m_s) \otimes \bigotimes_{q=1}^{m_s} \mathcal{P}(k_{g_s+q}) \right) & \xrightarrow{id \otimes (\bigotimes_{s=1}^n \gamma)} & \mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(h_s)
\end{array}$$

Equivariance. Let $n \geq 1$, let $k_1, \dots, k_n$ be non-negative integers and $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1}, \dots, \tau_n \in \Sigma_{k_n}$ permutations. Let $\sigma(k_1, \dots, k_n) \in \Sigma_{k_1 + \dots + k_n}$ denote the permutation that permutes n blocks $(1, \dots, k_1), \dots, (k_{n-1} + 1, \dots, k_n)$ as σ permutes $(1, \dots, n)$ and let $\tau_1 \oplus \dots \oplus \tau_n \in \Sigma_{k_1 + \dots + k_n}$ be the block sum of permutations. Then the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{P}(n) \otimes \mathcal{P}(k_{\sigma(1)}) \otimes \cdots \otimes \mathcal{P}(k_{\sigma(n)}) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(k_1 + \cdots + k_n) & \xrightarrow{\sigma(k_{\sigma(1)}, \dots, k_{\sigma(n)})} & \mathcal{P}(k_{\sigma(1)} + \cdots + k_{\sigma(n)})
\end{array}$$

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) & \xrightarrow{id \otimes \tau_1 \otimes \cdots \otimes \tau_n} & \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(k_1 + \cdots + k_n) & \xrightarrow{\tau_1 \oplus \cdots \oplus \tau_n} & \mathcal{P}(k_1 + \cdots + k_n)
\end{array}$$

Unitality. For each $n \geq 1$, the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathbf{k}^{\otimes n} & \xrightarrow{\cong} & \mathcal{P}(n) \\
id \otimes \eta^{\otimes n} \downarrow & \nearrow \gamma & \\
\mathcal{P}(n) \otimes \mathcal{P}(1)^{\otimes n} & &
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{k} \otimes \mathcal{P}(n) & \xrightarrow{\cong} & \mathcal{P}(n) \\
\eta \otimes id \downarrow & \nearrow \gamma & \\
\mathcal{P}(1) \otimes \mathcal{P}(n) & &
\end{array}$$

A straightforward modification of the above definition makes sense in any symmetric monoidal category $(\mathcal{M}, \odot, \mathbf{1})$ such as the category of differential graded modules, simplicial sets, topological spaces, etc, see [17] or [18]. We then speak about *differential graded operads*, *simplicial operads*, *topological operads*, etc.

EXAMPLE 3.1 *All properties axiomatized by Definition 3.1 can be read from the **endomorphism operad** $\text{End}_V = \{\text{End}_V(n)\}_{n \geq 0}$ of a $\mathbf{k}$ -module V . It is defined by setting $\text{End}_V(n)$ to be the space of $\mathbf{k}$ -linear maps $V^{\otimes n} \rightarrow V$. The operadic composition of $f \in \text{End}_V(n)$ with $g_1 \in \text{End}_V(k_1), \dots, g_n \in \text{End}_V(k_n)$ is given by the usual composition of multilinear maps as*

$$\gamma(f, g_1, \dots, g_n) := f(g_1 \otimes \cdots \otimes g_n),$$

the symmetric group acts by

$$\gamma\sigma(f, g_1, \dots, g_n) := f(g_{\sigma^{-1}(1)} \otimes \cdots \otimes g_{\sigma^{-1}(n)}), \quad \sigma \in \Sigma_n,$$

and the unit map $\eta : \mathbf{k} \rightarrow \text{End}_V(1)$ is given by $\eta(1) := \text{id}_V : V \rightarrow V$. The endomorphism operad can be constructed over an object of an arbitrary symmetric monoidal category with an internal hom-functor, as it was done in [17].

One often considers operads $\mathcal{A}$ such that $\mathcal{A}(0) = 0$ (the trivial $\mathbf{k}$ -module). We will indicate that $\mathcal{A}$ is of this type by writing $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 1}$.

EXAMPLE 3.2 *Let us denote by $\text{Ass} = \{\text{Ass}(n)\}_{n \geq 1}$ the operad with $\text{Ass}(n) := \mathbf{k}[\Sigma_n]$, $n \geq 1$, and the operadic composition defined as follows. Let $\text{id}_n \in \Sigma_n$, $\text{id}_{k_1} \in \Sigma_{k_1}, \dots, \text{id}_{k_n} \in \Sigma_{k_n}$ be the identity permutations. Then*

$$\gamma(\text{id}_n, \text{id}_{k_1}, \dots, \text{id}_{k_n}) := \text{id}_{k_1 + \dots + k_n} \in \Sigma_{k_1 + \dots + k_n}.$$

The above formula determines $\gamma(\sigma, \tau_1, \dots, \tau_n)$ for general $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1}, \dots, \tau_n \in \Sigma_{k_n}$ by the equivariance axiom. The unit map $\eta : \mathbf{k} \rightarrow \text{Ass}(1)$ is given by $\eta(1) := \text{id}_1$.

EXAMPLE 3.3 *Let us give an example of a topological operad. For $k \geq 1$, the **little k -discs operad** $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n \geq 0}$ is defined as follows [17]. Let*

$$\mathbb{D}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k; x_1^2 + \cdots + x_k^2 \leq 1\}$$

be the standard closed disc in $\mathbb{R}^k$. A little k -disc is then a linear embedding $d : \mathbb{D}^k \hookrightarrow \mathbb{D}^k$ which is the restriction of a linear map $\mathbb{R}^k \rightarrow \mathbb{R}^k$ with parallel axes. The n -th space $\mathcal{D}_k(n)$ of the little k -disc operad is the space of all n -tuples $(d_1, \dots, d_n)$ of little k -discs such that the images of $d_1, \dots, d_n$ have mutually disjoint interiors. The operad structure is obvious – the symmetric group Σ_n acts on $\mathcal{D}_k(n)$ by permuting the labels of the little discs and the structure map γ is given by composition of embeddings. The unit is the identity embedding $\text{id} : \mathbb{D}^k \hookrightarrow \mathbb{D}^k$.

EXAMPLE 3.4 The collection of normalized singular chains $C_*(\mathcal{T}) = \{C_*(\mathcal{T}(n))\}_{n \geq 0}$ of a topological operad $\mathcal{T} = \{\mathcal{T}(n)\}_{n \geq 0}$ is an operad in the category of differential graded $\mathbb{Z}$ -modules. For a ring R , the singular homology $H_*(\mathcal{T}(n); R) = H_*(C_*(\mathcal{T}(n)) \otimes_{\mathbb{Z}} R)$ forms an operad $H_*(\mathcal{T}; R)$ in the category of graded R -modules, see [16] for details.

DEFINITION 3.2 Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ and $\mathcal{Q} = \{\mathcal{Q}(n)\}_{n \geq 0}$ be two operads. A homomorphism $f : \mathcal{P} \rightarrow \mathcal{Q}$ is a sequence $f = \{f(n) : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)\}_{n \geq 0}$ of equivariant maps which commute with the operadic compositions and preserve the units.

An operad $\mathcal{R} = \{\mathcal{R}(n)\}_{n \geq 0}$ is a **suboperad** of $\mathcal{P}$ if $\mathcal{R}(n)$ is, for each $n \geq 0$, a Σ_n -submodule of $\mathcal{P}(n)$ and if all structure operations of $\mathcal{R}$ are the restrictions of those of $\mathcal{P}$. Finally, an ideal in the operad $\mathcal{P}$ is the collection $\mathcal{I} = \{\mathcal{I}(n)\}_{n \geq 0}$ of Σ_n -invariant subspaces $\mathcal{I}(n) \subset \mathcal{P}(n)$ such that

$$\gamma_{\mathcal{P}}(f, g_1, \dots, g_n) \in \mathcal{I}(k_1 + \dots + k_n)$$

if either $f \in \mathcal{I}(n)$ or $g_i \in \mathcal{I}(k_i)$ for some $1 \leq i \leq n$.

EXAMPLE 3.5 Given an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$, let $\widehat{\mathcal{P}} = \{\widehat{\mathcal{P}}(n)\}_{n \geq 0}$ be the collection defined by $\widehat{\mathcal{P}}(n) := \mathcal{P}(n)$ for $n \geq 1$ and $\widehat{\mathcal{P}}(0) := 0$. Then $\widehat{\mathcal{P}}$ is a suboperad of $\mathcal{P}$. The correspondence $\mathcal{P} \mapsto \widehat{\mathcal{P}}$ is a full embedding of the category of operads $\mathcal{P}$ with $\mathcal{P}(0) \cong \mathbf{k}$ into the category of operads $\mathcal{A}$ with $\mathcal{A}(0) = 0$. Operads satisfying $\mathcal{P}(0) \cong \mathbf{k}$ have been called **unital** while operads with $\mathcal{A}(0) = 0$ **non-unital** operads. We will not use this terminology because non-unital operads will mean something different in this book, see Section 4.

An example of an operad $\mathcal{A}$ which is not of the form $\widehat{\mathcal{P}}$ for some operad $\mathcal{P}$ with $\mathcal{P}(0) \cong \mathbf{k}$ can be constructed as follows. Observe first that operads $\mathcal{P}$ with the property that

$$\mathcal{P}(0) \cong \mathbf{k} \text{ and } \mathcal{P}(n) = 0 \text{ for } n \geq 2$$

are the same as augmented associative algebras. Indeed, the space $\mathcal{P}(1)$ with the operation $\circ_1 : \mathcal{P}(1) \otimes \mathcal{P}(1) \rightarrow \mathcal{P}(1)$ is clearly a unital associative algebra, augmented by the composition

$$\mathcal{P}(1) \xrightarrow{\cong} \mathcal{P}(1) \otimes \mathbf{k} \xrightarrow{\cong} \mathcal{P}(1) \otimes \mathcal{P}(0) \xrightarrow{\circ_1} \mathcal{P}(0) \cong \mathbf{k}.$$

Now take an arbitrary unital associative algebra A and define the operad $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 1}$ by

$$\mathcal{A}(n) := \begin{cases} A, & \text{for } n = 1 \text{ and} \\ 0, & \text{for } n \neq 1, \end{cases}$$

with $\circ_1 : \mathcal{A}(1) \otimes \mathcal{A}(1) \rightarrow \mathcal{A}(1)$ the multiplication of A . It follows from the above considerations that $\mathcal{A} = \widehat{\mathcal{P}}$ for some operad $\mathcal{P}$ with $\mathcal{P}(0) \cong \mathbf{k}$ if and only if A admits an augmentation. Therefore any unital associative algebra that does not admit an augmentation produces the desired example.

EXAMPLE 3.6 *Kernels, images, etc., of homomorphisms between operads in the category of $\mathbf{k}$ -modules are defined componentwise. For example, if $f : \mathcal{P} \rightarrow \mathcal{Q}$ is such homomorphism, then $\text{Ker}(f) = \{\text{Ker}(f)(n)\}_{n \geq 0}$ is the collection with*

$$\text{Ker}(f)(n) := \text{Ker}(f : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)), \quad n \geq 0.$$

It is clear that $\text{Ker}(f)$ is an ideal in $\mathcal{P}$.

Also quotients are defined componentwise. If $\mathcal{I}$ is an ideal in $\mathcal{P}$, then the collection $\mathcal{P}/\mathcal{I} = \{(\mathcal{P}/\mathcal{I})(n)\}_{n \geq 0}$ with $(\mathcal{P}/\mathcal{I})(n) := \mathcal{P}(n)/\mathcal{I}(n)$ for $n \geq 0$, has a natural operad structure induced by the structure of $\mathcal{P}$. The canonical projection $\mathcal{P} \rightarrow \mathcal{P}/\mathcal{I}$ has the expected universal property. The kernel of this projection equals $\mathcal{I}$.

Sometimes it suffices to consider operads without the symmetric group action. This notion is formalized by:

DEFINITION 3.3 (MAY'S NON- Σ OPERAD) *A **non- Σ operad** in the category of $\mathbf{k}$ -modules is a collection $\underline{\mathcal{P}} = \{\underline{\mathcal{P}}(n)\}_{n \geq 0}$ of $\mathbf{k}$ -modules, together with operadic compositions*

$$\underline{\gamma} : \underline{\mathcal{P}}(n) \otimes \underline{\mathcal{P}}(k_1) \otimes \cdots \otimes \underline{\mathcal{P}}(k_n) \rightarrow \underline{\mathcal{P}}(k_1 + \cdots + k_n),$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, and a unit map $\underline{\eta} : \mathbf{k} \rightarrow \underline{\mathcal{P}}(1)$ that fulfill the associativity and unitality axioms of Definition 3.1.

Each operad can be considered as a non- Σ operad by forgetting the Σ_n -actions. On the other hand, given a non- Σ operad $\underline{\mathcal{P}}$, there is an associated operad $\Sigma[\underline{\mathcal{P}}]$ with $\Sigma[\underline{\mathcal{P}}](n) := \underline{\mathcal{P}}(n) \otimes \mathbf{k}[\Sigma_n]$, $n \geq 0$, with the structure operations induced by the structure operations of $\underline{\mathcal{P}}$. Operads of this form are sometimes called *regular* operads.

EXAMPLE 3.7 Consider the operad $\mathcal{Com} = \{\mathcal{Com}(n)\}_{n \geq 1}$ such that $\mathcal{Com}(n) := \mathbf{k}$ with the trivial Σ_n -action, $n \geq 1$, and the operadic compositions (3.1) given by the canonical identifications

$$\mathcal{Com}(n) \otimes \mathcal{Com}(k_1) \otimes \cdots \otimes \mathcal{Com}(k_n) \cong \mathbf{k}^{\otimes(n+1)} \xrightarrow{\cong} \mathbf{k} \cong \mathcal{Com}(k_1 + \cdots + k_n).$$

The operad $\mathcal{Com}$ is obviously not regular. Observe also that $\mathcal{Com} \cong \widehat{\mathcal{End}}_{\mathbf{k}}$, where $\widehat{\mathcal{End}}_{\mathbf{k}}$ is the endomorphism operad of the ground ring without the initial component, see Example 3.5 for the notation.

Let $\underline{Ass}$ denote the operad $\mathcal{Com}$ considered as a non- Σ operad. Its symmetrization $\Sigma[\underline{Ass}]$ then equals the operad Ass introduced in Example 3.2.

As we already observed in Sections 1 and 2, there is an alternative approach to operads. For the purposes of comparison, in the rest of this Section and in the following Section we will refer to operads viewed from this alternative perspective as to Markl's operads.

DEFINITION 3.4 A **Markl's operad** in the category of $\mathbf{k}$ -modules is a collection $\mathcal{S} = \{\mathcal{S}(n)\}_{n \geq 0}$ of right $\mathbf{k}[\Sigma_n]$ -modules, together with $\mathbf{k}$ -linear maps ($\circ_i$ -compositions)

$$\circ_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \rightarrow \mathcal{S}(m + n - 1),$$

for $1 \leq i \leq m$ and $n \geq 0$. These data fulfill the following axioms.

Associativity. For each $1 \leq j \leq a$, $b, c \geq 0$, $f \in \mathcal{S}(a)$, $g \in \mathcal{S}(b)$ and $h \in \mathcal{S}(c)$,

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g, & \text{for } 1 \leq i < j, \\ f \circ_j (g \circ_{i-j+1} h), & \text{for } j \leq i < b+j, \text{ and} \\ (f \circ_{i-b+1} h) \circ_j g, & \text{for } j+b \leq i \leq a+b-1, \end{cases}$$

see Figure 10.

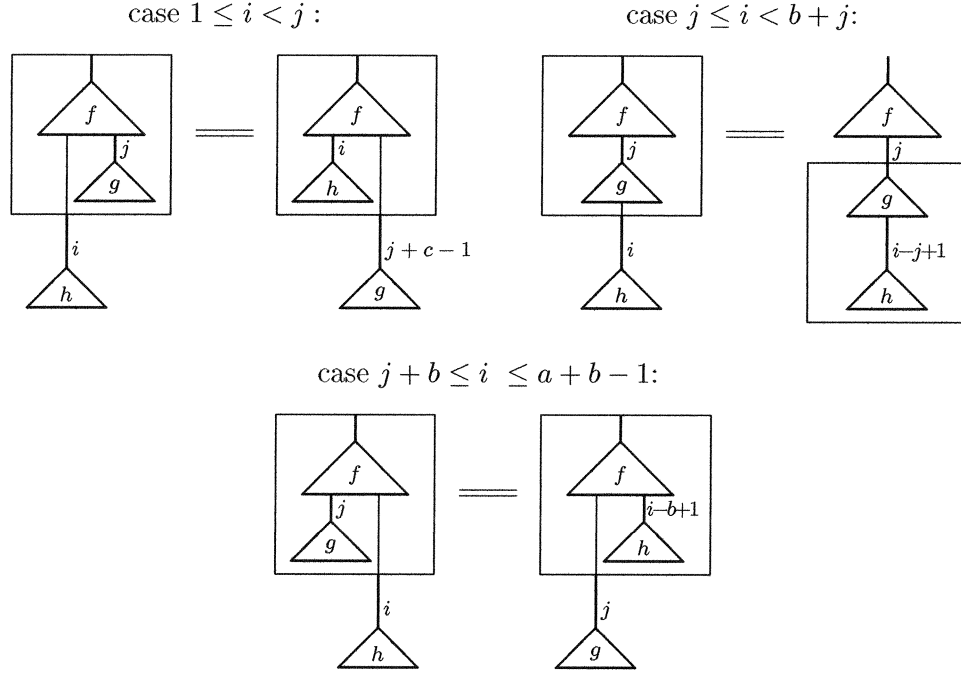


Figure 10: Flow charts explaining the associativity in Markl's operads.

Equivariance. For each $1 \leq i \leq m$, $n \geq 0$, $\tau \in \Sigma_m$ and $\sigma \in \Sigma_n$, let $\tau \circ_i \sigma \in \Sigma_{m+n-1}$ be given by inserting the permutation σ at the i th place in τ . Let $f \in \mathcal{S}(m)$ and $g \in \mathcal{S}(n)$. Then

$$(f\tau) \circ_i (g\sigma) = (f \circ_{\tau(i)} g)(\tau \circ_i \sigma).$$

Unitality. There exists $e \in \mathcal{S}(1)$ such that

$$f \circ_i e = e \quad \text{and} \quad e \circ_1 g = g \tag{3.2}$$

for each $1 \leq i \leq m$, $n \geq 0$, $f \in \mathcal{S}(m)$ and $g \in \mathcal{S}(n)$.

EXAMPLE 3.8 *All axioms in Definition 3.4 can be read from the endomorphism operad $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$ of a $\mathbf{k}$ -module V reviewed in Example 3.1, with $\circ_i$ -operations given by*

$$f \circ_i g := f(id_V^{\otimes i-1} \otimes g \otimes id_V^{\otimes m-1}),$$

for $f \in \mathcal{E}nd_V(m)$, $g \in \mathcal{E}nd_V(n)$, $1 \leq i \leq m$ and $n \geq 0$.

The following proposition shows that Definition 3.1 describes the same objects as Definition 3.4.

PROPOSITION 3.1 *The category of May's operads is isomorphic to the category of Markl's operads.*

Proof. Given a Markl's operad $\mathcal{S} = \{\mathcal{S}(n)\}_{n \geq 0}$ as in Definition 3.4, define a May's operad $\mathcal{P} = \text{May}(\mathcal{S})$ by $\mathcal{P}(n) := \mathcal{S}(n)$ for $n \geq 0$, with the γ -operations given by

$$\gamma(f, g_1, \dots, g_n) := (\cdots ((f \circ_n g_n) \circ_{n-1} g_{n-1}) \cdots) \circ_1 g_1 \quad (3.3)$$

where $f \in \mathcal{P}(n)$, $g_i \in \mathcal{P}(k_i)$, $1 \leq i \leq n$, $k_1, \dots, k_n \geq 0$. The unit morphism $\eta : \mathbf{k} \rightarrow \mathcal{P}(1)$ is defined by $\eta(1) := e$. It is easy to verify that $\text{May}(-)$ extends to a functor from the category of Markl's operads to the category of May's operads.

On the other hand, given a May's operad $\mathcal{P}$, one can define a Markl's operad $\mathcal{S} = \text{Mar}(\mathcal{P})$ by $\mathcal{S}(n) := \mathcal{P}(n)$ for $n \geq 0$, with the $\circ_i$ -operations:

$$f \circ_i g := \gamma(f, \underbrace{e, \dots, e}_{i-1}, g, \underbrace{e, \dots, e}_{m-i}), \quad (3.4)$$

for $f \in \mathcal{S}(m)$, $g \in \mathcal{S}(n)$, $m \geq 1$, $n \geq 0$, where $e := \eta(1) \in \mathcal{P}(1)$. It is again obvious that $\text{Mar}(-)$ extends to a functor that the functors $\text{May}(-)$ and $\text{Mar}(-)$ are mutually inverse isomorphisms between the category of Markl's operads and the category of May's operads. $\blacksquare$

The equivalence between May's and Markl's operads implies that an operad can be defined by specifying $\circ_i$ -operations and a unit. This is sometimes simpler than to define the γ -operations directly, as illustrated by:

EXAMPLE 3.9 *Let Σ be a Riemann sphere, that is, a nonsingular complex projective curve of genus 0. By a puncture or a parametrized hole we mean a point p of Σ together with a holomorphic embedding of the standard closed disc $U = \{z \in \mathbb{C}; |z| \leq 1\}$ to Σ centered at the point. Thus a puncture is a holomorphic embedding $u : \tilde{U} \rightarrow \Sigma$, where $\tilde{U} \subset \mathbb{C}$ is an open neighborhood of U and $u(0) = p$. We say that two punctures $u_1 : \tilde{U}_1 \rightarrow \Sigma$ and $u_2 : \tilde{U}_2 \rightarrow \Sigma$ are disjoint, if*

$$u_1(\overset{\circ}{U}) \cap u_2(\overset{\circ}{U}) = \emptyset,$$

where $\overset{\circ}{U} := \{z \in \mathbb{C}; |z| < 1\}$ is the interior of U .

Let $\widehat{\mathfrak{M}}_0(n)$ be the moduli space of Riemann spheres Σ with $n + 1$ disjoint punctures $u_i : \tilde{U}_i \rightarrow \Sigma$, $0 \leq i \leq n$, modulo the action of complex projective automorphisms. The topology of $\widehat{\mathfrak{M}}_0(n)$ is a very subtle thing and we are not going to discuss this issue here; see [19]. The constructions below will be made only ‘up to topology.’

Renumbering the holes $u_1, \dots, u_n$ defines on each $\widehat{\mathfrak{M}}_0(n)$ a natural right Σ_n -action and the Σ -module $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ forms a topological operad under sewing Riemannian spheres at punctures. Let us describe this operadic structure using the $\circ_i$ -formalism. Thus, let Σ represent an element $x \in \widehat{\mathfrak{M}}_0(m)$ and Δ represent an element $y \in \widehat{\mathfrak{M}}_0(n)$. For $1 \leq i \leq m$, let $u_i : \tilde{U}_i \rightarrow \Sigma$ be the i th puncture of Σ and let $u_0 : \tilde{U}_0 \rightarrow \Delta$ be the 0th puncture of Δ .

There certainly exists some $0 < r < 1$ such that both $\tilde{U}_0$ and $\tilde{U}_i$ contain the disc $U_{1/r} := \{z \in \mathbb{C}; |z| < 1/r\}$. Let now $\Sigma_r := \Sigma \setminus u_i(U_r)$ and $\Delta_r := \Delta \setminus u_0(U_r)$. Define finally

$$\Xi := (\Sigma_r \bigsqcup \Delta_r) / \sim,$$

where the relation $\sim$ is given by

$$\Sigma_r \ni u_i(\xi) \sim u_0(1/\xi) \in \Delta_r,$$

for $r < |\xi| < 1/r$. It is immediate to see that Ξ is a well-defined punctured Riemannian sphere, with $n + m - 1$ punctures induced in the obvious manner from those of Σ and Δ , and that the class of the punctured surface Ξ in the moduli space $\widehat{\mathfrak{M}}_0(m + n - 1)$ does not depend on the representatives Σ , Δ and on r . We define $x \circ_i y$ to be the class of Ξ .

The unit $e \in \widehat{\mathfrak{M}}_0(1)$ can be defined as follows. Let $\mathbb{CP}^1$ be the complex projective line with homogeneous coordinates $[z, w]$, $z, w \in \mathbb{C}$, [20, Example I.1.6]. Let $0 := [0, 1] \in \mathbb{CP}^1$ and $\infty := [1, 0] \in \mathbb{CP}^1$. Recall that we have two canonical isomorphisms $p_\infty : \mathbb{CP}^1 \setminus \infty \rightarrow \mathbb{C}$ and $p_0 : \mathbb{CP}^1 \setminus 0 \rightarrow \mathbb{C}$ given by

$$p_\infty([z, w]) := z/w \text{ and } p_0([z, w]) := w/z.$$

Then $p_\infty^{-1} : \mathbb{C} \rightarrow \mathbb{CP}^1$ (respectively $p_0^{-1} : \mathbb{C} \rightarrow \mathbb{CP}^1$) is a puncture at 0 (respectively at ∞). We define $e \in \widehat{\mathfrak{M}}_0(1)$ to be the class of $(\mathbb{CP}^1, p_0^{-1}, p_\infty^{-1})$.

It is not hard to verify that the above constructions make the collection $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ a Markl’s operad. By Proposition 3.1, $\widehat{\mathfrak{M}}_0$ is also May’s operad.

In the rest of this book, we will consider May's and Markl's operads as two versions of the same object which we will call simply a *unital operad*.

4 Non-unital Operads

It turns out that the combinatorial structure of the moduli space of stable genus zero curves is captured by a certain non-unital version of operad. Let $\mathcal{M}_{0,n+1}$ be the moduli space of $(n+1)$ -tuples $(x_0, \dots, x_n)$ of distinct numbered points on the complex projective line $\mathbb{CP}^1$ modulo projective automorphisms, that is, transformations of the form

$$\mathbb{CP}^1 \ni [\xi_1, \xi_2] \mapsto [a\xi_1 + b\xi_2, c\xi_1 + d\xi_2] \in \mathbb{CP}^1,$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

The moduli space $\mathcal{M}_{0,n+1}$ has, for $n \geq 2$, a canonical compactification $\overline{\mathcal{M}}_0(n) \supset \mathcal{M}_{0,n+1}$ introduced by A. Grothendieck and F.F. Knudsen [21, 22]. The space $\overline{\mathcal{M}}_0(n)$ is the moduli space of stable $(n+1)$ -pointed curves of genus 0:

DEFINITION 4.1 A **stable $(n+1)$ -pointed curve of genus 0** is an object

$$(C; x_0, \dots, x_n),$$

where C is a (possibly reducible) algebraic curve with at most nodal singularities and $x_0, \dots, x_n \in C$ are distinct smooth points such that

- (i) each component of C is isomorphic to $\mathbb{CP}^1$,
- (ii) the graph of intersections of components of C (i.e. the graph whose vertices correspond to the components of C and edges to the intersection points of the components) is a tree, and
- (iii) each component of C has at least three special points, where a special point means either one of the x_i , $0 \leq i \leq n$, or a singular point of C (the stability).

It can be seen that a stable curve $(C; x_0, \dots, x_n)$ admits no infinitesimal automorphisms that fix marked points $x_0, \dots, x_n$, therefore $(C; x_0, \dots, x_n)$ is 'stable' in the usual sense. Observe also that $\overline{\mathcal{M}}_0(0) = \overline{\mathcal{M}}_0(1) = \emptyset$ (there are no stable curves

with less than three marked points) and that $\overline{\mathcal{M}}_0(2)$ = the point corresponding to the three-pointed stable curve $(\mathbb{CP}^1; \infty, 1, 0)$. The space $\mathcal{M}_{0,n+1}$ forms an open dense part of $\overline{\mathcal{M}}_0(n)$ consisting of marked curves $(C; x_0, \dots, x_n)$ such that C is isomorphic to $\mathbb{CP}^1$.

Let us try to equip the collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ with an operad structure as in Definition 3.1. For $C = (C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_0(n)$ and $C_i = (C_i, y_1^i, \dots, y_{k_i}^i) \in \overline{\mathcal{M}}_0(k_i)$, $1 \leq i \leq n$, let

$$\gamma(C, C_1, \dots, C_n) \in \overline{\mathcal{M}}_0(k_1 + \dots + k_n) \quad (4.1)$$

be the stable marked curve obtained from the disjoint union $C \sqcup C^1 \sqcup \dots \sqcup C^n$ by identifying, for each $1 \leq i \leq n$, the point $x_i \in C$ with the point $y_0^i \in C_i$, introducing a nodal singularity, and relabeling the remaining marked points accordingly. The symmetric group acts on $\overline{\mathcal{M}}_0(n)$ by

$$(C, x_0, x_1, \dots, x_n) \mapsto (C, x_0, x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \sigma \in \Sigma_n.$$

We have defined the γ -compositions and the symmetric group action, but there is no room for the identity, because $\overline{\mathcal{M}}_0(1)$ is empty! The above structure is, therefore, a non-unital operad in the sense of the following definition (which is formulated, as all definitions in Sections 1.3–1.11, for the monoidal category of $\mathbf{k}$ -modules).

DEFINITION 4.2 *A **May's non – unital operad** in the category of $\mathbf{k}$ -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of $\mathbf{k}[\Sigma_n]$ -modules, together with operadic compositions*

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \rightarrow \mathcal{P}(k_1 + \dots + k_n),$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, that fulfill the associativity and equivariance axioms of Definition 3.1.

We may as well define on the collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ operations

$$\circ_i : \overline{\mathcal{M}}_0(m) \times \overline{\mathcal{M}}_0(n) \rightarrow \overline{\mathcal{M}}_0(m + n - 1) \quad (4.2)$$

for $m, n \geq 2$, $1 \leq i \leq m$, by

$$(C^1; y_0, \dots, y_m) \times (C^2; x_0, \dots, x_n) \mapsto (C; y_0, \dots, y_{i-1}, x_0, \dots, x_n, y_{i+1}, \dots, y_m)$$

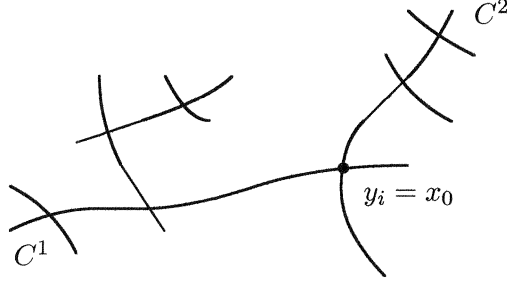


Figure 11: The $\circ_i$ -compositions in $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$.

where C is the quotient of the disjoint union $C^1 \sqcup C^2$ given by identifying x_0 with y_i at a nodal singularity, see Figure 11. The collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ with $\circ_i$ -operations (4.2) is an example of another version of non-unital operads, recalled in:

DEFINITION 4.3 *A **non – unital Markl’s operad** in the category of $\mathbf{k}$ -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of $\mathbf{k}[\Sigma_n]$ -modules, together with operadic compositions*

$$\circ_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \rightarrow \mathcal{S}(m + n - 1),$$

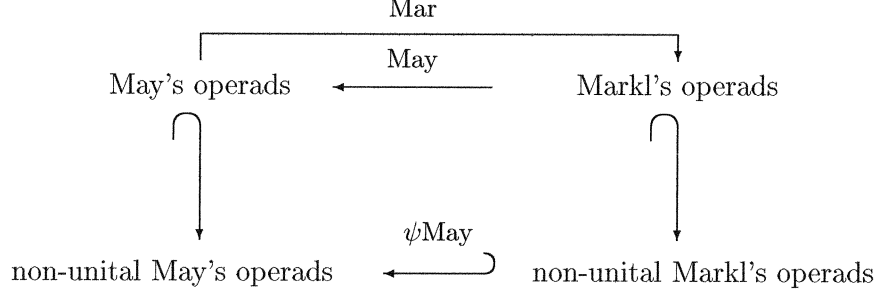
for $1 \leq i \leq m$ and $n \geq 0$, that fulfill the associativity and equivariance axioms of Definition 3.4.

As we saw in Proposition 3.1, in the presence of operadic units, May’s operads are the same as Markl’s operads. Surprisingly, the non-unital versions of these structures are *radically different* – Markl’s operads capture more information than May’s operads! This is made precise in the following:

PROPOSITION 4.1 *The category of non-unital Markl’s operads is a subcategory of the category of non-unital May’s operads.*

Proof. It is easy to see that (3.3) defines, as in the proof of Proposition 3.1, a functor $\psi\text{May}(-)$ which is an embedding of the category of non-unital Markl’s operads into the category of non-unital May’s operads. ■

Observe that formula (3.4), inverse to (3.3), does not make sense without units. The relation between various versions of operads discussed so far is summarized in the following diagram of categories and their inclusions:



The following example shows that non-unital Markl's operads form a proper sub-category of the category of non-unital May's operads.

EXAMPLE 4.1 *We describe a non-unital May's operad $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$ which is not of the form $\psi \text{May}(\mathcal{S})$ for some non-unital Markl's operad $\mathcal{S}$. Let*

$$\mathcal{V}(n) := \begin{cases} \mathbf{k}, & \text{for } n = 2 \text{ or } 4, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

The only non-trivial γ -composition is $\gamma : \mathcal{V}(2) \otimes \mathcal{V}(2) \otimes \mathcal{V}(2) \rightarrow \mathcal{V}(4)$, given as the canonical isomorphism

$$\mathcal{V}(2) \otimes \mathcal{V}(2) \otimes \mathcal{V}(2) \cong \mathbf{k}^{\otimes 3} \xrightarrow{\cong} \mathbf{k} \cong \mathcal{V}(4).$$

Suppose that $\mathcal{V} = \text{May}(\mathcal{S})$ for some non-unital Markl's operad $\mathcal{S}$. Then, according to (3.3), for $f, g_1, g_2 \in \mathcal{V}(2)$,

$$\gamma(f, g_1, g_2) = (f \circ_2 g_2) \circ_1 g_1.$$

Since $(f \circ_2 g) \in \mathcal{V}(3) = 0$, this would imply that γ is trivial, which is not true.

Proposition 4.2 below shows that Markl's rather than May's non-unital operads are true non-unital versions of operads. We will need the following definition in which $\mathcal{K} = \{\mathcal{K}(n)\}_{n \geq 1}$ is the trivial (unital) operad with $\mathcal{K}(1) := \mathbf{k}$ and $\mathcal{K}(n) = 0$, for $n \neq 1$.

DEFINITION 4.4 An **augmentation** of an operad $\mathcal{P}$ in the category of $\mathbf{k}$ -modules is a homomorphism $\epsilon : \mathcal{P} \rightarrow \mathcal{K}$. Operads with an augmentation are called **augmented operads**. The kernel

$$\overline{\mathcal{P}} := \text{Ker}(\epsilon : \mathcal{P} \rightarrow \mathcal{K})$$

is called the **augmentation ideal**.

The following proposition was proved in [23].

PROPOSITION 4.2 The correspondence $\mathcal{P} \mapsto \overline{\mathcal{P}}$ is an isomorphism between the category of augmented operads and the category of Markl's non-unital operads.

Proof. The $\circ_i$ -operations of $\mathcal{P}$ obviously restrict to $\overline{\mathcal{P}}$, making it a non-unital Markl's operad. It is simple to describe a functorial inverse $\mathcal{S} \mapsto \tilde{\mathcal{S}}$ of the correspondence $\mathcal{P} \mapsto \overline{\mathcal{P}}$. For a Markl's non-unital operad $\mathcal{S}$, denote by $\tilde{\mathcal{S}}$ the collection

$$\tilde{\mathcal{S}}(n) := \begin{cases} \mathcal{S}(n), & \text{for } n \neq 1, \text{ and} \\ \mathcal{S}(1) \oplus \mathbf{k} & \text{for } n = 1. \end{cases} \quad (4.3)$$

The $\circ_i$ -operations of $\tilde{\mathcal{S}}$ are uniquely determined by requiring that they extend the $\circ_i$ -operations of $\mathcal{S}$ and satisfy (3.2), with the unit $e := 0 \oplus 1_{\mathbf{k}} \in \mathcal{S}(1) \oplus \mathbf{k} = \tilde{\mathcal{S}}(1)$. Informally, $\tilde{\mathcal{S}}$ is obtained from the Markl's non-unital operad $\mathcal{S}$ by adjoining a unit. $\blacksquare$

Observe that if $\mathcal{S}$ were a May's, not Markl's, non-unital operad, the construction of $\tilde{\mathcal{S}}$ described in the above proof would not make sense, because we would not know how to define

$$\gamma(f, \underbrace{e, \dots, e}_{i-1}, g, \underbrace{e, \dots, e}_{m-i})$$

for $f \in \mathcal{S}(m)$, $g \in \mathcal{S}(n)$, $m \geq 2$, $n \geq 0$, $1 \leq i \leq m$. Proposition 4.2 should be compared to the obvious statement that the category of augmented unital associative algebras is isomorphic to the category of (non-unital) associative algebras. In the following proposition, $\mathbf{Oper}$ denotes the category of $\mathbf{k}$ -linear operads and $\psi\mathbf{Oper}$ the category of $\mathbf{k}$ -linear Markl's non-unital operads.

PROPOSITION 4.3 Let $\mathcal{P}$ be an augmented operad and $\mathcal{Q}$ an arbitrary operad in the category of $\mathbf{k}$ -modules. Then there exists a natural isomorphism

$$\text{Mor}_{\mathbf{Oper}}(\mathcal{P}, \mathcal{Q}) \cong \text{Mor}_{\psi\mathbf{Oper}}(\overline{\mathcal{P}}, \psi\text{May}(\mathcal{Q})). \quad (4.4)$$

The proof is simple and we leave it to the reader. Combining (4.4) with the isomorphism of Proposition 4.2 one obtains a natural isomorphism

$$\text{Mor}_{\mathbf{Oper}}(\tilde{\mathcal{S}}, \mathcal{Q}) \cong \text{Mor}_{\psi\mathbf{Oper}}(\mathcal{S}, \psi\text{May}(\mathcal{Q})) \quad (4.5)$$

which holds for each Markl's non-unital operad $\mathcal{S}$ and operad $\mathcal{Q}$. Isomorphism (4.5) means that $\tilde{\cdot} : \psi\mathbf{Oper} \rightarrow \mathbf{Oper}$ and $\psi\text{May} : \mathbf{Oper} \rightarrow \psi\mathbf{Oper}$ are adjoint functors. This adjunction will be used in the construction of free operads in Section 6.

In the rest of this book, non-unital Markl's operads will be called simply non-unital operads. This will not lead to confusion, since all non-unital operads referred to in the rest of this book will be Markl's.

5 Operad Algebras

As we already remarked, operads are important through their representations called operad algebras or simply algebras.

DEFINITION 5.1 *Let V be a $\mathbf{k}$ -module and End_V the endomorphism operad of V recalled in Example 3.1. A $\mathcal{P}$ -algebra is a homomorphism of operads $\rho : \mathcal{P} \rightarrow \text{End}_V$.*

The above definition admits an obvious generalization into an arbitrary symmetric monoidal category with an internal hom-functor. The last assumption is necessary for the existence of the 'internal' endomorphism operad, see [17]. Definition 5.1 can be however unwrapped into the form given in [16] that makes sense in an arbitrary symmetric monoidal category without the internal hom-functor assumption:

PROPOSITION 5.1 *Let $\mathcal{P}$ be an operad. A $\mathcal{P}$ -algebra is the same as a $\mathbf{k}$ -module V together with maps*

$$\alpha : \mathcal{P}(n) \otimes V^{\otimes n} \rightarrow V, \quad n \geq 0, \quad (5.1)$$

that satisfy the following axioms.

Associativity. For each $n \geq 1$ and non-negative integers $k_1, \dots, k_n$, the following diagram commutes.

$$\begin{array}{ccc}
\left(\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(k_s) \right) \otimes \bigotimes_{s=1}^n V^{\otimes k_s} & \xrightarrow{\gamma \otimes id} & \mathcal{P}(k_1 + \dots + k_n) \otimes V^{\otimes(k_1 + \dots + k_n)} \\
\downarrow \text{shuffle} & & \downarrow \alpha \\
\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \left(\mathcal{P}(k_s) \otimes V^{\otimes k_s} \right) & \xrightarrow{id \otimes (\bigotimes_{s=1}^n \alpha)} & \mathcal{P}(n) \otimes V^{\otimes n} \\
& & \uparrow \alpha \\
& & V
\end{array}$$

Equivariance. For each $n \geq 1$ and $\sigma \in \Sigma_n$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes V^{\otimes n} & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{P}(n) \otimes V^{\otimes n} \\
& \searrow \alpha & \swarrow \alpha \\
& V &
\end{array}$$

Unitality. For each $n \geq 1$, the following diagram commutes.

$$\begin{array}{ccc}
\mathbf{k} \otimes V & \xrightarrow{\cong} & V \\
\eta \otimes id \downarrow & \nearrow \alpha & \\
\mathcal{P}(1) \otimes V & &
\end{array}$$

We leave as an exercise to formulate a version of Proposition 5.1 that would use $\circ_i$ -operations instead of γ -operations.

EXAMPLE 5.1 In this example we verify, using Proposition 5.1, that algebras over the operad $\text{Com} = \{\text{Com}(n)\}_{n \geq 1}$ recalled in Example 3.7 are ordinary commutative associative algebras. To simplify the exposition, let us agree that v 's with various subscripts denote elements of V . Since $\text{Com}(n) = \mathbf{k}$ for $n \geq 1$, the structure map (5.1) determines, for each $n \geq 1$, a linear map $\mu_n : V^{\otimes n} \rightarrow V$ by

$$\mu_n(v_1, \dots, v_n) := \alpha(1_n, v_1, \dots, v_n),$$

where 1_n denotes in this example the unit $1_n \in \mathbf{k} = \text{Com}(n)$. The associativity of Proposition 5.1 says that

$$\begin{aligned} \mu_n(\mu_{k_1}(v_1, \dots, v_{k_1}), \dots, \mu_{k_n}(v_{k_1+\dots+k_{n-1}+1}, \dots, v_{k_1+\dots+k_n})) = \\ \mu_{k_1+\dots+k_n}(v_1, \dots, v_{k_1+\dots+k_n}), \end{aligned} \quad (5.2)$$

for each $n, k_1, \dots, k_n \geq 1$. The equivariance of Proposition 5.1 means that each μ_n is fully symmetric

$$\mu_n(v_1, \dots, v_n) = \mu_n(v_{\sigma(1)}, \dots, v_{\sigma(n)}), \quad \sigma \in \Sigma_n, \quad (5.3)$$

and the unitality implies that μ_1 is the identity map,

$$\mu_1(v) = v. \quad (5.4)$$

The above structure can be identified with a commutative associative multiplication on V . Indeed, the bilinear map $\cdot := \mu_2 : V \otimes V \rightarrow V$ is clearly associative:

$$(v_1 \cdot v_2) \cdot v_3 = v_1 \cdot (v_2 \cdot v_3) \quad (5.5)$$

and commutative:

$$v_1 \cdot v_2 = v_2 \cdot v_1. \quad (5.6)$$

On the other hand, $\mu_1(v) := v$ and

$$\mu_n(v_1, \dots, v_n) := (\dots(v_1 \cdot v_2) \cdots v_{n-1}) \cdot v_n \quad \text{for } n \geq 2$$

defines multilinear maps $\{\mu_n : V^{\otimes n} \rightarrow V\}$ satisfying (5.2)–(5.4). It is equally easy to verify that algebras over the operad Ass introduced in Example 3.2 are ordinary associative algebras.

Following Leinster [14], one could say that (5.2)–(5.4) is an *unbiased* definition of associative commutative algebras, while (5.5)–(5.6) is a definition of the same object *biased* towards bilinear operations. Operads therefore provide unbiased definitions of algebras.

EXAMPLE 5.2 Let us denote by UCom the endomorphism operad $\text{End}_{\mathbf{k}}$ of the ground ring $\mathbf{k}$. It is easy to verify that UCom -algebras are **unital** commutative associative algebras. We leave it to the reader to describe the operad UAss governing unital associative operads.

Algebras over a non- Σ operad $\mathcal{P}$ are defined as algebras, in the sense of Definition 5.1, over the symmetrization $\Sigma[\mathcal{P}]$ of $\mathcal{P}$. Algebras over non-unital operads discussed in Section 4 are defined by appropriate obvious modifications of Definition 5.1.

EXAMPLE 5.3 *Let Y be a topological space with a base point $*$ and $\mathbb{S}^k$ the k -dimensional sphere, $k \geq 1$. The k -fold loop space $\Omega^k Y$ is the space of all continuous maps $\mathbb{S}^k \rightarrow Y$ that send the south pole of $\mathbb{S}^k$ to the base point of Y . Equivalently, $\Omega^k Y$ is the space of all continuous maps $\lambda : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *)$ from the standard closed k -dimensional disc $\mathbb{D}^k$ to Y that map the boundary $\mathbb{S}^{k-1}$ of $\mathbb{D}^k$ to the base point of Y . Let us show, following Boardman and Vogt [24], that $\Omega^k Y$ is a natural topological algebra over the little k -discs operad $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n \geq 0}$ recalled in Example 3.3.*

*The action $\alpha : \mathcal{D}_k(n) \times (\Omega^k Y)^{\times n} \rightarrow \Omega^k Y$ is, for $n \geq 0$, defined as follows. Given $\lambda_i : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *) \in \Omega^k Y$, $1 \leq i \leq n$, and little k -discs $d = (d_1, \dots, d_n) \in \mathcal{D}_k(n)$ as in Example 3.3, then*

$$\alpha(d, \lambda_1, \dots, \lambda_n) : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *) \in \Omega^k Y$$

is the map defined to be $\lambda_i : \mathbb{D}^k \rightarrow Y$ (suitably rescaled) on the image of d_i , and to be $$ on the complement of the images of the maps d_i , $1 \leq i \leq n$.*

Therefore each k -fold loop space is a $\mathcal{D}_k$ -space. The following classical theorem is a certain form of the inverse statement.

THEOREM 5.1 (Boardman-Vogt [24], May [25]) *A path-connected $\mathcal{D}_k$ -algebra X has the weak homotopy type of a k -fold loop space.*

The connectedness assumption in the above theorem can be weakened by assuming that the $\mathcal{D}_k$ -action makes the set $\pi_0(X)$ of path components of X a group.

EXAMPLE 5.4 *The non-unital operad $\overline{\mathcal{M}}_0$ of stable pointed curves of genus 0 (also called the **configuration (non – unital) operad**) recalled on page 42 is a non-unital operad in the category of smooth complex projective varieties. It therefore makes sense, as explained in Example 3.4, to consider its homology operad $H_*(\overline{\mathcal{M}}_0, \mathbf{k}) = \{H_*(\overline{\mathcal{M}}_0(n), \mathbf{k})\}_{n \geq 2}$.*

An algebra over this non-unital operad is called a (tree level) **cohomological conformal field theory** or a **hyper – commutative algebra** [10]. It consist of a family of linear operations $\{(-, \dots, -) : V^{\otimes n} \rightarrow V\}_{n \geq 2}$ which are totally symmetric, that is

$$(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (v_1, \dots, v_n),$$

for each permutation $\sigma \in \Sigma_n$. Moreover, we require the following form of associativity:

$$\sum_{(S,T)} ((u, v, x_i; i \in S), w, x_j; j \in T) = \sum_{(S,T)} (u, (v, w, x_i; i \in S), x_j; j \in T), \quad (5.7)$$

where $u, v, w, x_1, \dots, x_n \in V$ and (S, T) runs over disjoint decompositions $S \sqcup T = \{1, \dots, n\}$. For $n = 0$, (5.7) means the (usual) associativity of the bilinear operation $(-, -)$, i.e. $((u, v), w) = (u, (v, w))$. For $n = 1$ we get

$$((u, v), w, x) + ((u, v, x), w) = (u, (v, w, x)) + (u, (v, w), x).$$

EXAMPLE 5.5 In this example, $\mathbf{k}$ is a field of characteristic 0. The non-unital operad $\overline{\mathcal{M}}_0(\mathbb{R}) = \{\overline{\mathcal{M}}_0(\mathbb{R})(n)\}_{n \geq 2}$ of real points in the configuration operad $\overline{\mathcal{M}}_0$ is called the **mosaic non – unital operad** [26]. Algebras over the homology $H_*(\overline{\mathcal{M}}_0(\mathbb{R}), \mathbf{k}) = \{H_*(\overline{\mathcal{M}}_0(\mathbb{R})(n), \mathbf{k})\}_{n \geq 2}$ of this operad were recently identified [27] with **2-Gerstenhaber algebras**, which are structures (V, μ, τ) consisting of a commutative associative product $\mu : V \otimes V \rightarrow V$ and an anti-symmetric degree +1 ternary operation $\tau : V \otimes V \otimes V \rightarrow V$ which satisfies the generalized Jacobi identity

$$\sum_{\sigma} \text{sgn}(\sigma) \cdot \tau(\tau(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), x_{\sigma(4)}, x_{\sigma(5)}) = 0,$$

where the summation runs over all (3, 2)-unshuffles $\sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(4) < \sigma(5)$. Moreover, the ternary operation τ is tied to the multiplication μ by the distributive law

$$\tau(\mu(s, t), u, v) = \mu(\tau(s, u, v), t) + (-1)^{(1+|u|+|v|)|s|} \cdot \mu(s, \tau(t, u, v)), \quad s, t, u, v \in V,$$

saying that the assignment $s \mapsto \tau(s, u, v)$ is a degree $(1 + |u| + |v|)$ -derivation of the associative commutative algebra (V, μ) , for each $u, v \in V$.

6 Free Operads and a Category of rooted Trees

The purpose of this Section is three-fold. First, we want to study free operads because each operad is a quotient of a free one. The second reason why we are interested in free operads is that their construction involves trees. Indeed, it turns out that rooted trees provide ‘pasting schemes’ for operads and that, replacing trees by other types of graphs, one can introduce several important generalizations of operads, such as cyclic operads, modular operads, and PROPs. The last reason is that the free operad functor defines a monad which provides an unbiased definition of operads as algebras over this monad. Everything in this Section is written for $\mathbf{k}$ -linear operads, but the constructions can be generalized into an arbitrary symmetric monoidal category with countable coproducts $(\mathcal{M}, \odot, \mathbf{1})$ whose monoidal product $\odot$ is distributive over coproducts, see [17].

Recall that a Σ -module is a collection $E = \{E(n)\}_{n \geq 0}$ in which each $E(n)$ is a right $\mathbf{k}[\Sigma_n]$ -module. There is an obvious forgetful functor $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$ from the category $\mathbf{Oper}$ of $\mathbf{k}$ -linear operads to the category $\Sigma\text{-mod}$ of Σ -modules.

DEFINITION 6.1 *The **free operad functor** is a left adjoint [28] $\Gamma : \Sigma\text{-mod} \rightarrow \mathbf{Oper}$ to the forgetful functor $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$. This means that there exists a functorial isomorphism*

$$\text{Mor}_{\mathbf{Oper}}(\Gamma(E), \mathcal{P}) \cong \text{Mor}_{\Sigma\text{-mod}}(E, \square(\mathcal{P}))$$

*for an arbitrary Σ -module E and operad $\mathcal{P}$. The operad $\Gamma(E)$ is the free operad generated by the Σ -module E . Similarly, the **free non-unital operad functor** is a left adjoint $\Psi : \Sigma\text{-mod} \rightarrow \psi\mathbf{Oper}$ of the obvious forgetful functor $\square_\psi : \psi\mathbf{Oper} \rightarrow \Sigma\text{-mod}$, that is*

$$\text{Mor}_{\psi\mathbf{Oper}}(\Psi(E), \mathcal{S}) \cong \text{Mor}_{\Sigma\text{-mod}}(E, \square_\psi(\mathcal{S})),$$

*where E is a Σ -module and $\mathcal{S}$ a non-unital operad. The non-unital operad $\Psi(E)$ is the **free non-unital operad** generated by the Σ -module E .*

Let $\tilde{\cdot} : \psi\mathbf{Oper} \rightarrow \mathbf{Oper}$ be the functor of ‘adjoining the unit’ considered in the proof of Proposition 4.2 on page 45. Functorial isomorphism (4.5) implies that one may take

$$\Gamma := \tilde{\Psi}, \tag{6.1}$$

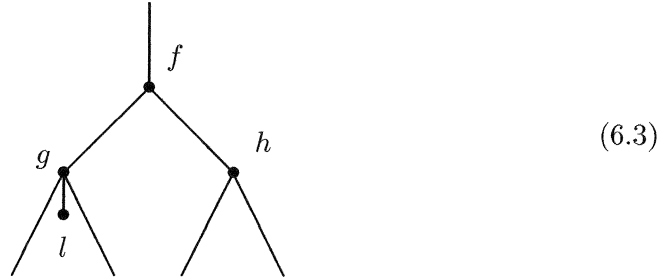
which means that the free operad $\Gamma(E)$ can be obtained from the free non-unital operad $\Psi(E)$ by formally adjoining the unit.

Let us indicate how to construct the free non-unital operad $\Psi(E)$, a precise description will be given later in this Section. The free non-unital operad $\Psi(E)$ must be built up from all formal $\circ_i$ -compositions of elements of E modulo the axioms listed in Definition 3.4. For instance, given $f \in E(2)$, $g \in E(3)$, $h \in E(2)$ and $l \in E(0)$, the component $\Psi(E)(5)$ must contain the following five compositions

$$\begin{aligned} & (f \circ_1 (g \circ_2 l)) \circ_3 h, \quad (f \circ_2 h) \circ_1 (g \circ_2 l), \quad ((f \circ_2 h) \circ_1 g) \circ_2 l, \\ & ((f \circ_1 g) \circ_2 l) \circ_3 h \quad \text{and} \quad ((f \circ_1 g) \circ_4 h) \circ_2 l. \end{aligned} \tag{6.2}$$

The elements in (6.2) can be depicted by the ‘flow diagrams’ of Figure 12. Nodes of these diagrams are decorated by elements f, g, h and l of E in such a way that an element of $E(n)$ decorates a node with n input lines, $n \geq 0$. Thin ‘amoebas’ indicate the nesting which specifies the order in which the $\circ_i$ -operations are performed.

The associativity of Definition 3.4 however says that the result of the composition does not depend on the order, therefore the amoebas can be erased and the common value of the compositions represented by



Let us look more closely how diagram (6.3) determines an element of the (still hypothetical) free non-unital operad $\Psi(E)$. The crucial fact is that the underlying graph of (6.3) is a planar rooted tree. Recall (see Section 2) that a *tree* is a finite connected simply connected graph without loops and multiple edges. For a tree T we denote, as usual, by $Vert(T)$ the set of vertices and $Edg(T)$ the set of edges of T . The number of edges adjacent to a vertex $v \in Vert(T)$ is called the *valence* of v and denoted $val(v)$. We assume that one is given a subset

$$ext(T) \subset \{v \in Vert(T); \quad val(v) = 1\}$$

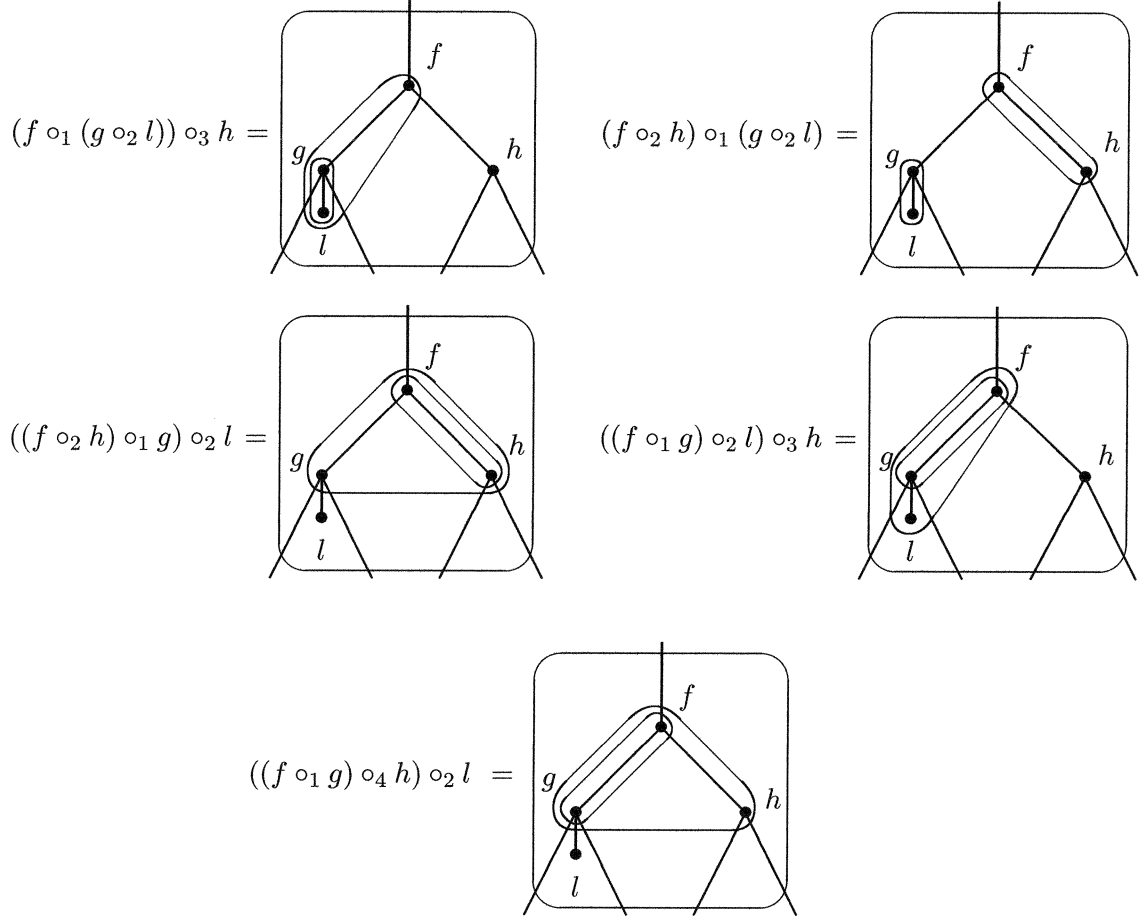


Figure 12: Flow diagrams in non-unital operads.

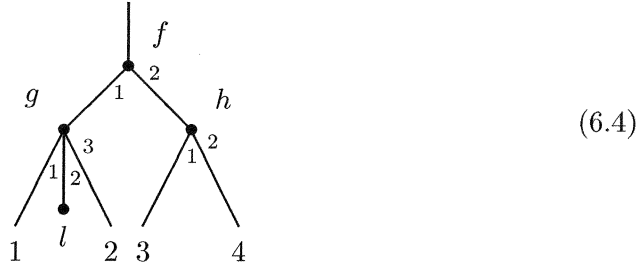
of *external* vertices, the remaining vertices are *internal*. Let us denote

$$\text{vert}(T) := \text{Vert}(T) \setminus \text{ext}(T)$$

the set of all internal vertices. Henceforth, we will assume that our trees have at least one internal vertex. This excludes at this stage the *exceptional tree* consisting of two external vertices connected by an edge.

Edges adjacent to external vertices are the *legs* of T . A tree is *rooted* if one of its legs, called the *root*, is marked and all other edges are oriented, pointing to the root. The legs different from the root are the *leaves* of T . For example, the tree in (6.3) has 4 internal vertices decorated f , g , h and l , and 4 leaves. Finally, the *planarity* means that an embeddings of T into the plane is specified. In Sections 6–11 for all pictures, the root will always be placed on the top. By a vertex we will always mean an internal one.

The planarity and a choice of the root of the underlying tree of (6.3) specifies a total order of the set $\text{in}(v)$ of input edges of each vertex $v \in \text{vert}(T)$ as well as a total order of the set $\text{Leaf}(T)$ of the leaves of T , by numbering from the left to the right:



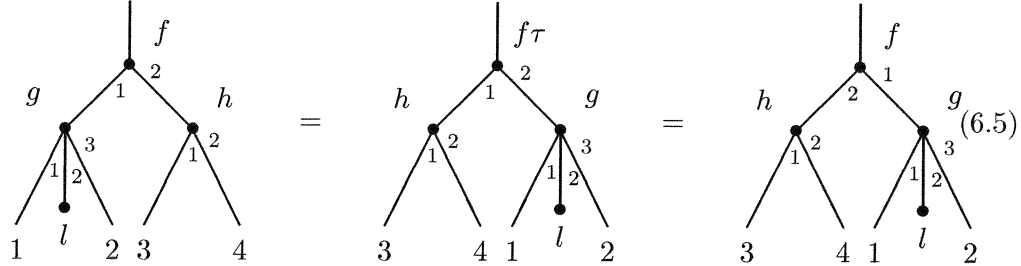
This tells us that l should be inserted into the second input of g , g into the first input of f and h into the second input of f . Using ‘abstract variables’ v_1, v_2, v_3 and v_4 , the element represented by (6.4) can also be written as the ‘composition’ $f(g(v_1, l, v_2), h(v_3, v_4))$.

Now we need to take into account also the symmetric group action. If τ is the generator of Σ_2 , then the obvious equality

$$f(g(v_1, l, v_2), h(v_3, v_4)) = f\tau(h(v_3, v_4), g(v_1, l, v_2))$$

of ‘abstract compositions’ coming from the equivariance of Definition 3.4 trans-

lates into the following equality of flow diagrams:



Relation (6.5) shows that the equivariance of Definition 3.4 violates the linear orders induced by the planar embedding of T . This leads us to the conclusion that the flow diagrams describing elements of free non-unital operads are (abstract, non-planar) rooted, leaf-labeled decorated trees.

Let us describe, after these motivations, a precise construction of $\Psi(E)$. The first subtlety one needs to understand is how to decorate vertices of non-planar trees. In Sections 6–11, we need to explain how each Σ -module $E = \{E(n)\}_{n \geq 0}$ naturally extends into a functor (denoted again E) from the category $\mathbf{Set}_f$ of finite sets and their bijections to the category of $\mathbf{k}$ -modules. If X and Y are finite sets, denote by

$$Bij(Y, X) := \{\vartheta : X \xrightarrow{\cong} Y\} \quad (6.6)$$

the set of all isomorphisms between X and Y (notice the unexpected direction of the arrow!). It is clear that $Bij(Y, X)$ is a natural left Aut_Y -right Aut_X -bimodule, where $Aut_X := Bij(X, X)$ and $Aut_Y := Bij(Y, Y)$ are the sets of automorphisms with group structure given by composition. For a finite set $S \in \mathbf{Set}_f$ of cardinality n and a Σ -module $E = \{E(n)\}_{n \geq 0}$ define $E(S)$ to be

$$E(S) := E(n) \times_{\Sigma_n} \text{Bij}([n], S) \quad (6.7)$$

where, as usual, $[n] := \{1, \dots, n\}$ and, of course, $\Sigma_n = \text{Aut}_{[n]}$.

Let us recall that a *(leaf-) labeled rooted n -tree* is a rooted tree T together with a specified bijection $\ell : \text{Leaf}(T) \xrightarrow{\sim} [n]$. Let $\mathbf{Tree}_n$ be the category of labeled rooted n -trees and their bijections. For $T \in \mathbf{Tree}_n$ define

$$E(T) := \bigotimes_{v \in \text{vert}(T)} E(\text{in}(v)) \quad (6.8)$$

where $in(v)$ is, as before, the set of all input edges of a vertex $v \in vert(T)$. It is easy to verify that $E \mapsto E(T)$ defines a functor from the category $\mathbf{Tree}_n$ to the category of $\mathbf{k}$ -modules.

Recall that the colimit of a covariant functor $F : \mathcal{D} \rightarrow \mathbf{Mod}_{\mathbf{k}}$ is the quotient

$$\operatorname{colim}_{x \in \mathcal{D}} F(x) = \bigoplus_{x \in \mathcal{D}} F(x) / \sim,$$

where $\sim$ is the equivalence generated by

$$F(y) \ni a \sim F(f)(a) \in F(z),$$

for each $a \in F(y)$, $y, z \in \mathcal{D}$ and $f \in Mor_{\mathcal{D}}(y, z)$. Define finally

$$\Psi(E)(n) := \operatorname{colim}_{T \in \mathbf{Tree}_n} E(T), \quad n \geq 0. \quad (6.9)$$

The following theorem was proved in [17].

THEOREM 6.1 *There exists a natural non-unital operad structure on the Σ -module*

$$\Psi(E) = \{\Psi(E)(n)\}_{n \geq 0},$$

with the $\circ_i$ -operations given by the grafting of trees and the symmetric group re-labeling the leaves, such that $\Psi(E)$ is the free non-unital operad generated by the Σ -module E .

One could simplify (6.9) by introducing $\mathcal{T}ree(n)$ as the set of *isomorphism* classes of n -trees from $\mathbf{Tree}_n$ and defining $\Psi(E)$ by the formula

$$\Psi(E)(n) = \bigoplus_{[T] \in \mathcal{T}ree(n)} E(T), \quad n \geq 0, \quad (6.10)$$

which does not involve the colimit. The drawback of (6.10) is that it assumes a choice of a representative $[T]$ of each isomorphism class in $\mathcal{T}ree(n)$, while (6.9) is functorial and admits simple generalizations to other types of operads and PROPs. See [17] for other representations of the free non-unital operad functor.

Having constructed the free non-unital operad $\Psi(E)$, we may use (6.1) to define the free operad $\Gamma(E)$. This is obviously equivalent to enlarging, in (6.9) for

$n = 1$, the category $\mathbf{Tree}_n$ by the *exceptional rooted tree* $\begin{array}{c} | \\ \hline \end{array}$ with one leg and no internal vertex. If we denote this enlarged category of trees and their isomorphisms (which however differs from $\mathbf{Tree}_n$ only at $n = 1$) by $\mathbf{UTree}_n$, we may represent the free operad as

$$\Gamma(E)(n) := \operatorname{colim}_{T \in \mathbf{UTree}_n} E(T), \quad n \geq 0. \quad (6.11)$$

If E is a Σ -module such that $E(0) = E(1) = 0$, then (6.10) reduces to a summation over *reduced trees*, that is trees whose all vertices have at least two input edges. By simple combinatorics, the number of isomorphism classes of reduced trees in $\mathbf{Tree}_n$ is finite for each $n \geq 0$. This implies the following proposition that says that operads are relatively small objects.

PROPOSITION 6.1 *Let $E = \{E(n)\}_{n \geq 0}$ be a Σ -module such that*

$$E(0) = E(1) = 0$$

and that $E(n)$ are finite-dimensional for $n \geq 2$. Then the spaces $\Psi(E)(n)$ and $\Gamma(E)(n)$ are finite-dimensional for each $n \geq 0$.

We close this Section by showing how the free operad functor can be used to define operads. It follows from general principles that any operad $\mathcal{P}$ is a quotient $\mathcal{P} = \Gamma(E)/(R)$, where E and R are Σ -modules and (R) is the operadic ideal (see Definition 3.2) generated by R in $\Gamma(E)$.

EXAMPLE 6.1 *The commutative associative operad $\mathcal{Com}$ recalled in Example 3.7 is generated by the Σ -module*

$$E_{\mathcal{Com}}(n) := \begin{cases} \mathbf{k} \cdot \mu, & \text{if } n = 2 \text{ and} \\ 0, & \text{if } n \neq 2. \end{cases}$$

where $\mathbf{k} \cdot \mu$ is the trivial representation of Σ_2 . The ideal of relations is generated by

$$R_{\mathcal{Com}} := \operatorname{Span}_{\mathbf{k}}\{\mu(\mu \otimes id) - \mu(id \otimes \mu)\} \subset \Gamma(E_{\mathcal{Com}})(3),$$

where $\mu(\mu \otimes id) - \mu(id \otimes \mu)$ is the obvious shorthand for $\gamma(\mu, \mu, e) - \gamma(\mu, e, \mu)$, with e the unit of $\Gamma(E_{\mathcal{Com}})$.

Similarly, the operad $\mathcal{A}ss$ for associative algebras reviewed in Example 3.2 is generated by the Σ -module $E_{\mathcal{A}ss}$ such that

$$E_{\mathcal{A}ss}(n) := \begin{cases} \mathbf{k}[\Sigma_2], & \text{if } n = 2 \text{ and} \\ 0, & \text{if } n \neq 2. \end{cases}$$

The ideal of relations is generated by the $\mathbf{k}[\Sigma_3]$ -closure $R_{\mathcal{A}ss}$ of the associativity

$$\alpha(\alpha \otimes id) - \alpha(id \otimes \alpha) \in \Gamma(E_{\mathcal{A}ss})(3), \quad (6.12)$$

where α is a generator of the regular representation $E_{\mathcal{A}ss}(2) = \mathbf{k}[\Sigma_2]$.

EXAMPLE 6.2 The operad $\mathcal{L}ie$ governing Lie algebras is the quotient $\mathcal{L}ie := \Gamma(E_{\mathcal{L}ie})/(R_{\mathcal{L}ie})$, where $E_{\mathcal{L}ie}$ is the Σ -module

$$E_{\mathcal{L}ie}(n) := \begin{cases} \mathbf{k} \cdot \beta, & \text{if } n = 2 \text{ and} \\ 0, & \text{if } n \neq 2, \end{cases}$$

with $\mathbf{k} \cdot \beta$ is the signum representation of Σ_2 . The ideal of relations $(R_{\mathcal{L}ie})$ is generated by the Jacobi identity:

$$\beta(\beta \otimes id) + \beta(\beta \otimes id)c + \beta(\beta \otimes id)c^2 = 0, \quad (6.13)$$

in which $c \in \Sigma_3$ is the cyclic permutation $(1, 2, 3) \mapsto (2, 3, 1)$.

EXAMPLE 6.3 We show how to describe the presentations of the operads $\mathcal{A}ss$ and $\mathcal{L}ie$ given in Examples 6.1 and 6.2 in a simple graphical language. The generator α of $E_{\mathcal{A}ss}$ is an operation with two inputs and one output, so we depict it as $\wedge$. The associativity (6.12) then reads as

$$\wedge = \wedge,$$

therefore $\mathcal{A}ss = \Gamma(\wedge)/(\wedge = \wedge)$. Also the operad for $\mathcal{L}ie$ algebras is generated by one bilinear operation $\wedge$, but this time the operation is anti-symmetric

$$\begin{array}{c} \wedge \\ 1 \quad 2 \end{array} = - \begin{array}{c} \wedge \\ 2 \quad 1 \end{array}.$$

The Jacobi identity (6.13) reads

$$\begin{array}{c} \wedge \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} \wedge \\ 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} \wedge \\ 3 \quad 1 \quad 2 \end{array} = 0.$$

The kind of description used in the above examples is ‘tautological’ in the sense that it just says that the operad $\mathcal{P}$ governing a certain type of algebras is generated by operations of these algebras, with an appropriate symmetry, modulo the axioms satisfied by these operations. It does not say directly anything about the properties of the individual spaces $\mathcal{P}(n)$, $n \geq 0$. Describing these individual components may be a very nontrivial task, see for example the formula for the Σ_n -modules $\mathcal{L}ie(n)$ given in [17]. Operads in Examples 6.1 and 6.2 are quadratic in the sense of the following:

DEFINITION 6.2 *An operad $\mathcal{P}$ is **quadratic** if it has a presentation $\mathcal{P} = \Gamma(E)/(R)$, where $E = \mathcal{P}(2)$ and $R \subset \Gamma(E)(3)$.*

Quadratic operads form a very important class of operads. Each quadratic operad $\mathcal{P}$ has a *quadratic dual* $\mathcal{P}^!$ [29], [17] which is a quadratic operad defined, roughly speaking, by dualizing the generators of $\mathcal{P}$ and replacing the relations of $\mathcal{P}$ by their annihilator in the dual space. For example, $\mathcal{A}ss^! = \mathcal{A}ss$, $\mathcal{C}om^! = \mathcal{L}ie$ and $\mathcal{L}ie^! = \mathcal{C}om$. A quadratic operad $\mathcal{P}$ is *Koszul* if it has the homotopy type of the bar construction of its quadratic dual [29], [17]. For quadratic Koszul operads, there is a deep understanding of the derived category of the corresponding algebras. Operads $\mathcal{A}ss$, $\mathcal{C}om$ and $\mathcal{L}ie$ above, as well as most quadratic operads one encounters in everyday life, are Koszul.

7 Category of May’s Trees

In this Section, we review the definition of a triple (monad) and give, in Theorem 7.1, a description of unital and non-unital operads in terms of algebras over a triple. The relevant triples come from the endofunctors Ψ and Γ recalled in Section 6. Let $End(\mathcal{C})$ be the strict symmetric monoidal category of endofunctors on a category $\mathcal{C}$ where multiplication is the composition of functors.

DEFINITION 7.1 *A **triple** (also called a **monad**) T on a category $\mathcal{C}$ is an associative and unital monoid (T, μ, ν) in $End(\mathcal{C})$. The multiplication $\mu : TT \rightarrow T$ and unit morphism $\nu : id \rightarrow T$ satisfy the axioms given by commutativity of the diagrams in Figure 13.*

$$\begin{array}{ccccc}
& & T\mu & & \\
& & \longrightarrow & & \\
TTT & & TT & & \\
\downarrow \mu T & & \downarrow \mu & & \\
TT & \xrightarrow{\mu} & T & &
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{Tv} & TT \\
\downarrow id & & \downarrow \mu \\
& T &
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{vT} & TT \\
\downarrow id & & \downarrow \mu \\
& T &
\end{array}$$

Figure 13: Associativity and unit axioms for a triple.

Triples arise naturally from pairs of adjoint functors. Given an adjoint pair [28, II.7]

$$\begin{array}{ccc}
& G & \\
A & \xrightarrow{\quad} & B \\
& F &
\end{array}$$

with associated functorial isomorphism

$$Mor_A(F(X), Y) \cong Mor_B(X, G(Y)), \quad X \in B, \quad Y \in A,$$

there is a triple in B defined by $T := GF$. The unit of the adjunction $id \rightarrow GF$ defines the unit v of the triple and the counit of the adjunction $FG \rightarrow id$ induces a natural transformation $GFGF \rightarrow GF$ which defines the multiplication μ . In fact, it is a theorem of Eilenberg and Moore [30] that all triples arise in this way from adjoint pairs. This is exactly the situation with the free operad and free non-unital operad functors that were described in Section 6. We will show how operads and non-unital operads can actually be defined using the concept of an algebra over a triple:

DEFINITION 7.2 *A T -algebra or **algebra over the triple** T is an object A of $\mathcal{C}$ together with a structure morphism $\alpha : T(A) \rightarrow A$ satisfying*

$$\alpha(T(\alpha)) = \alpha(\mu_A) \text{ and } \alpha v_A = id_A,$$

see Figure 14.

The category of T -algebras in $\mathcal{C}$ will be denoted $\mathbf{Alg}_T(\mathcal{C})$. Since the free non-unital operad functor Ψ and the free operad functor Γ described in Section 6 are left adjoints to $\square_\psi : \psi\mathbf{Oper} \rightarrow \Sigma\text{-mod}$ and $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$, respectively, the functors $\square_\psi \Psi$ (denoted simply Ψ) and $\square \Gamma$ (denoted Γ) define triples on $\Sigma\text{-mod}$.

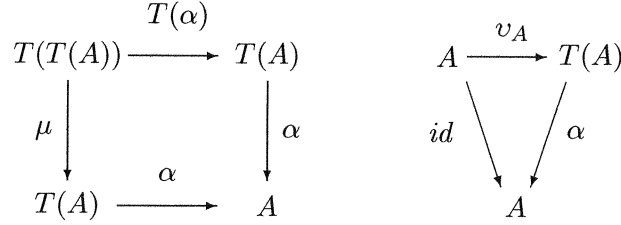
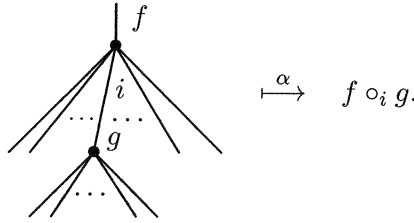


Figure 14: T -algebra structure.

THEOREM 7.1 *A Σ -module $\mathcal{S}$ is a Ψ -algebra if and only if it is a non-unital operad and it is a Γ -algebra if and only if it is an operad. In shorthand:*

$$\mathrm{Alg}_{\Psi}(\Sigma\text{-mod}) \cong \psi\mathrm{Oper} \quad \text{and} \quad \mathrm{Alg}_{\Gamma}(\Sigma\text{-mod}) \cong \mathrm{Oper}.$$

Proof. We outline first the proof of the implication in the direction from algebra to non-unital operad. Let $\mathcal{S}$ be a Ψ -algebra. The restriction of the structure morphism $\alpha : \Psi(\mathcal{S}) \longrightarrow \mathcal{S}$ to the components of $\Psi(\mathcal{S})$ supported on trees with one internal edge defines the non-unital operad composition maps $\circ_i$, as indicated by:



In the opposite direction, for a non-unital operad $\mathcal{S}$, the Ψ -algebra structure $\alpha : \Psi(\mathcal{S}) \rightarrow \mathcal{S}$ is the contraction along the edges of underlying trees, using the $\circ_i$ -operations. The proof that Γ -algebras are operads is similar. $\blacksquare$

Let us change our perspective and consider formula (6.9) as defining an endofunctor $\Psi : \Sigma\text{-mod} \rightarrow \Sigma\text{-mod}$, ignoring that we already know that it represents free non-unital operads. We are going to construct maps

$$\mu : \Psi\Psi \rightarrow \Psi \text{ and } v : id \rightarrow \Psi$$

making Ψ a triple on the category $\Sigma\text{-mod}$. Let us start with the triple multiplication μ . It follows from (6.9) that, for each Σ -module E ,

$$\Psi\Psi(E)(n) := \operatorname{colim}_{T \in \mathbf{Tree}_n} \Psi(E)(T), \quad n \geq 0.$$

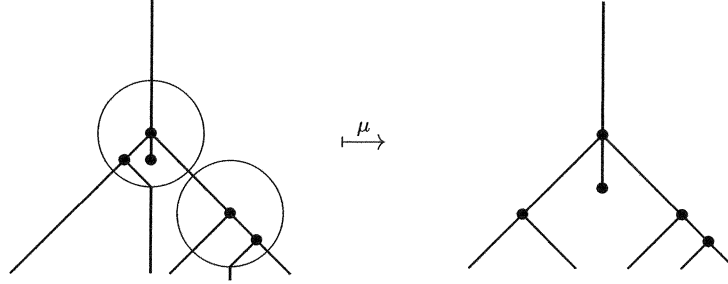


Figure 15: Bracketed trees. The left picture shows an element of $\Psi\Psi(E)(5)$ while the right picture shows the same element interpreted, after erasing the braces indicated by thin cycles, as an element of $\Psi(E)(5)$. For simplicity, we did not show the decoration of vertices by elements of E .

The elements in the right hand side are represented by rooted trees T with vertices decorated by elements of $\Psi(E)$, while elements of $\Psi(E)$ are represented by rooted trees with vertices decorated by E . We may therefore imagine elements of $\Psi\Psi(E)$ as ‘bracketed’ rooted trees, in the sense indicated in Figure 15. The triple multiplication $\mu_E : \Psi\Psi(E) \rightarrow \Psi(E)$ then simply erases the braces. The triple unit $\nu_E : E \rightarrow \Psi(E)$ identifies elements of E with decorated corollas:

$$E(n) \ni e \quad \longleftrightarrow \quad \begin{array}{c} \text{---} e \\ \bullet \\ \diagup \quad \dots \quad \diagdown \\ \underbrace{\hspace{1.5cm}} \\ n \text{ inputs} \end{array} \in \Psi(E)(n), \quad n \geq 0.$$

It is not difficult to verify that the above constructions indeed make Ψ a triple, compare [17]. Now we can define non-unital operads as algebras over the triple (Ψ, μ, ν) . The advantage of this approach is that, by replacing $\mathbf{Tree}_n$ in (6.9) by another category of trees or graphs, one may obtain triples defining other types of operads and their generalizations.

We have already seen in (6.11) that enlarging $\mathbf{Tree}_n$ into $\mathbf{UTree}_n$ by adding the exceptional tree, one gets the triple Γ describing (unital) operads. It is not difficult to see that non-unital May’s operads are related to the category $\mathbf{MTree}_n$ of *May’s trees* which are, by definition, rooted trees whose vertices can be arranged

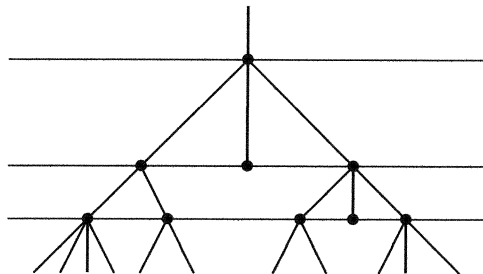


Figure 16: A May's tree.

into levels as in Figure 16. Non-unital May's operads are then algebras over the triple $M : \Sigma\text{-mod} \rightarrow \Sigma\text{-mod}$ defined by

$$M(E)(n) := \operatorname{colim}_{T \in \mathbf{MTree}_n} E(T), \quad n \geq 0.$$

These observations are summarized in the first three lines of the table in Figure 23 on page 89.

8 Cyclic Operads and non-rooted Trees

In the following two Sections we use the approach developed in Section 7 to introduce cyclic and modular operads. We recalled, in Example 3.9, the operad $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ of Riemann spheres with parametrized labeled holes. Each $\widehat{\mathfrak{M}}_0(n)$ was a right Σ_n -space, with the operadic right Σ_n -action permuting the labels $1, \dots, n$ of the holes $u_1, \dots, u_n$. But each $\widehat{\mathfrak{M}}_0(n)$ obviously admits a higher type of symmetry which interchanges labels $0, \dots, n$ of *all* holes, including the label of the 'output' hole u_0 . Another example admitting a similar higher symmetry is the configuration (non-unital) operad $\overline{\mathfrak{M}}_0 = \{\overline{\mathfrak{M}}_0(n)\}_{n \geq 2}$.

These examples indicate that, for some operads, there is no clear distinction between 'inputs' and the 'output.' Cyclic operads, introduced by E. Getzler and M.M. Kapranov in [9], formalize this phenomenon. They are, roughly speaking, operads with an extra symmetry that interchanges the output with one of the inputs. Let us recall some notions necessary to give a precise definition.

We remind the reader that in this Section, as well as everywhere in Sections 3–11, main definitions are formulated over the underlying category of $\mathbf{k}$ -modules, where $\mathbf{k}$ is a commutative associative unital ring. However as in Sections 1–2, for some constructions, we will require $\mathbf{k}$ to be a *field*; we will indicate this as usual by speaking about *vector spaces* instead of $\mathbf{k}$ -modules.

Let Σ_n^+ be the permutation group of the set $\{0, \dots, n\}$. The group Σ_n^+ is, of course, non-canonically isomorphic to the symmetric group Σ_{n+1} . We identify Σ_n with the subgroup of Σ_n^+ consisting of permutations $\sigma \in \Sigma_n^+$ such that $\sigma(0) = 0$. If $\tau_n \in \Sigma_n^+$ denotes the cycle $(0, \dots, n)$, that is, the permutation with $\tau_n(0) = 1$, $\tau_n(1) = 2$, $\dots$, $\tau_n(n) = 0$, then τ_n and Σ_n generate Σ_n^+ .

Recall that a *cyclic Σ -module* or a Σ^+ -*module* is a sequence $W = \{W(n)\}_{n \geq 0}$ such that each $W(n)$ is a (right) $\mathbf{k}[\Sigma_n^+]$ -module. Let $\Sigma^+\text{-mod}$ denote the category of cyclic Σ -modules. As (ordinary) operads were Σ -modules with an additional structure, cyclic operads are Σ^+ -modules with an additional structure.

We will also need the following ‘cyclic’ analog of (6.7): if X is a set with $n + 1$ elements and $W \in \Sigma^+\text{-mod}$, then

$$W(\langle X \rangle) := W(n) \times_{\Sigma_n^+} \text{Bij}([n]^+, X), \quad (8.1)$$

where $[n]^+ := \{0, \dots, n\}$, $n \geq 0$. Double brackets in $W(\langle X \rangle)$ remind us that the n th piece of the cyclic Σ -module $W = \{W(n)\}_{n \geq 0}$ is applied on a set with $n + 1$ elements, using the extended Σ_n^+ -symmetry. Therefore

$$W(\langle \{0, \dots, n\} \rangle) \cong W(n) \quad \text{while} \quad W(\{0, \dots, n\}) \cong W(n + 1), \quad n \geq 0.$$

Pasting schemes for cyclic operads are *cyclic (leg-) labeled n -trees*, by which we mean non-rooted trees, with legs labeled by the set $\{0, \dots, n\}$. An example of such a tree is given in Figure 17. Since we do not assume a choice of the root, the edges of a cyclic tree C are not directed and it does not make sense to speak about inputs and the output of a vertex $v \in \text{vert}(C)$. Let $\mathbf{Tree}_n^+$ be the category of cyclic labeled n -trees and their bijections.

For a cyclic Σ -module W and a cyclic labeled tree T we have the following cyclic version of the product (6.8)

$$W(\langle T \rangle) := \bigotimes_{v \in \text{vert}(T)} W(\langle \text{edge}(v) \rangle).$$

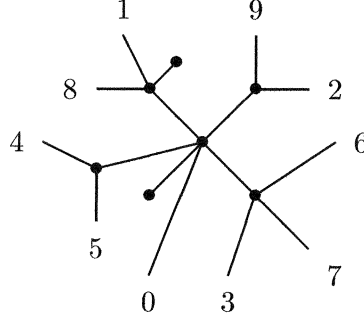


Figure 17: A cyclic labeled tree from $\mathbf{Tree}_9^+$.

The conceptual difference between (6.8) and the above formula is that instead of the set $\text{in}(v)$ of incoming edges of a vertex v of a rooted tree, here we use the set $\text{edge}(v)$ of *all* edges incident with v . Let, finally, $\Psi_+ : \Sigma^+\text{-mod} \rightarrow \Sigma^+\text{-mod}$ be the functor

$$\Psi_+(W)(n) := \operatorname{colim}_{T \in \mathbf{Tree}_n^+} W(\langle T \rangle), \quad n \geq 0, \quad (8.2)$$

equipped with the triple structure of ‘forgetting the braces’ similar to that reviewed on page 62. We will use also the ‘extended’ triple $\Gamma_+ : \Sigma^+\text{-mod} \rightarrow \Sigma^+\text{-mod}$,

$$\Gamma_+(W)(n) := \operatorname{colim}_{T \in \mathbf{UTree}_n^+} W(\langle T \rangle), \quad n \geq 0,$$

where $\mathbf{UTree}_n^+$ is the obvious extension of the category $\mathbf{Tree}_n^+$ by the exceptional tree $\mid$.

DEFINITION 8.1 A ***cyclic*** (resp. ***non – unital cyclic***) ***operad*** is an algebra over the triple Γ_+ (resp. the triple Ψ_+) introduced above.

In the following proposition, which slightly improves [9], $\tau_n \in \Sigma_n^+$ denotes the cycle $(0, \dots, n)$.

PROPOSITION 8.1 A non-unital cyclic operad is the same as a non-unital operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 0}$ (Definition 3.4) such that the right Σ_n -action on $\mathcal{C}(n)$ extends, for each $n \geq 0$, to an action of Σ_n^+ with the property that for $p \in \mathcal{C}(m)$ and $q \in \mathcal{C}(n)$,

$1 \leq i \leq m, n \geq 0$, the composition maps satisfy

$$(p \circ_i q) \tau_{m+n-1} = \begin{cases} (q \tau_n) \circ_n (p \tau_m), & \text{if } i = 1, \text{ and} \\ (p \tau_m) \circ_{i-1} q, & \text{for } 2 \leq i \leq m. \end{cases}$$

The above structure is a (unital) cyclic operad if moreover there exists a Σ_1^+ -invariant operadic unit $e \in \mathcal{C}(1)$.

Proposition 8.1 gives a biased definition of cyclic operads whose obvious modification (see [17]) makes sense in an arbitrary symmetric monoidal category. We can therefore speak about topological cyclic operads, differential graded cyclic operads, simplicial cyclic operads etc. Observe that there are no *non-unital cyclic May's operads* because it does not make sense to speak about levels in trees without a choice of the root.

EXAMPLE 8.1 Let V be a finite dimensional vector space and $B : V \otimes V \rightarrow \mathbf{k}$ a nondegenerate symmetric bilinear form. The form B induces the identification

$$\text{Lin}(V^{\otimes n}, V) \ni f \mapsto \widehat{B}(f) := B(-, f(-)) \in \text{Lin}(V^{\otimes(n+1)}, \mathbf{k})$$

of the spaces of linear maps. The standard right Σ_n^+ -action

$$\widehat{B}(f) \sigma(v_0, \dots, v_n) = \widehat{B}(f)(v_{\sigma^{-1}(0)}, \dots, v_{\sigma^{-1}(n)}), \quad \sigma \in \Sigma_n^+, \quad v_0, \dots, v_n \in V,$$

defines, via this identification, a right Σ_n^+ -action on $\text{Lin}(V^{\otimes n}, V)$, that is, on the n th piece of the endomorphism operad $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$ recalled in Example 3.1. It is easy to show that, with the above action, $\mathcal{E}nd_V$ is a cyclic operad in the monoidal category of vector spaces, called the **cyclic endomorphism operad** of the pair $V = (V, B)$. The biased definition of cyclic operads given in Proposition 8.1 can be read off from this example.

EXAMPLE 8.2 We saw in Example 3.5 that a unital operad $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 0}$ such that $\mathcal{A}(n) = 0$ for $n \neq 1$ is the same as a unital associative algebra. Similarly, it can be easily shown that a cyclic operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 0}$ satisfying $\mathcal{C}(n) = 0$ for $n \neq 1$ is the same as a unital associative algebra A with a linear involutive antiautomorphism, by which we mean a $\mathbf{k}$ -linear map $^* : A \rightarrow A$ such that

$$(ab)^* = b^* a^*, \quad (a^*)^* = a \quad \text{and} \quad 1^* = 1,$$

for arbitrary $a, b \in A$.

Let $\mathcal{P} = \Gamma(E)/(R)$ be a quadratic operad as in Definition 6.2. The action of Σ_2 on E extends to an action of Σ_2^+ , via the sign representation $\text{sgn} : \Sigma_2^+ \rightarrow \{\pm 1\} = \Sigma_2$. It can be easily verified that this action induces a cyclic operad structure on the free operad $\Gamma(E)$. In particular, $\Gamma(E)(3)$ is a right Σ_3^+ -module.

DEFINITION 8.2 *We say that the operad $\mathcal{P}$ is a **cyclic quadratic operad** if, in the above presentation, R is a Σ_3^+ -invariant subspace of $\Gamma(E)(3)$.*

If the condition of the above definition is satisfied, $\mathcal{P}$ has a natural induced cyclic operad structure.

EXAMPLE 8.3 *By [9], all quadratic operads generated by a one-dimensional space are cyclic quadratic, therefore the operads $\mathcal{L}ie$ and $\mathcal{C}om$ are cyclic quadratic. Also the operads $\mathcal{A}ss$ and the operad $\mathcal{P}oiss$ for Poisson algebras are cyclic quadratic [9]. A surprisingly simple operad which is cyclic and quadratic, but not cyclic quadratic, is constructed in [63].*

The operad $\widehat{\mathfrak{M}}_0$ of Riemann spheres with labeled punctures reviewed in Example 3.9 is a topological cyclic operad. The configuration operad $\overline{\mathfrak{M}}_0$ recalled on page 42 is a non-unital topological cyclic operad. Important examples of non-cyclic operads are the operad $pre\text{-}\mathcal{L}ie$ for pre-Lie algebras [63, Section 3] and the operad $\mathcal{L}eib$ for Leibniz algebras [9].

Let $\mathcal{C}$ be an operad, $\alpha : \mathcal{C}(n) \otimes V^{\otimes n} \rightarrow V$, $n \geq 0$, a $\mathcal{C}$ -algebra with the underlying vector space V as in Proposition 5.1 and $B : V \otimes V \rightarrow U$ a bilinear form on V with values in a vector space U . We can form a map

$$\tilde{B}(\alpha) : \mathcal{C}(n) \otimes V^{\otimes(n+1)} \rightarrow U, \quad n \geq 0, \quad (8.3)$$

by the formula

$$\tilde{B}(\alpha)(c \otimes v_0 \otimes \cdots \otimes v_n) := B(v_0, \alpha(c \otimes v_1 \otimes \cdots \otimes v_n)), \quad c \in \mathcal{C}(n), \quad v_0, \dots, v_n \in V.$$

Suppose now that the operad $\mathcal{C}$ is cyclic, in particular, that each $\mathcal{C}(n)$ is a right Σ_n^+ -module. We say that the bilinear form $B : V \otimes V \rightarrow U$ is *invariant* [9], if the maps $\tilde{B}(\alpha)$ in (8.3) are, for each $n \geq 0$, invariant under the diagonal action of Σ_n^+ on $\mathcal{C}(n) \otimes V^{\otimes(n+1)}$. We leave as an exercise to verify that the invariance of

$\tilde{B}(\alpha)$ for $n = 1$ together with the existence of the operadic unit implies that B is symmetric,

$$B(v_0, v_1) = B(v_1, v_0), \quad v_0, v_1 \in V.$$

DEFINITION 8.3 *A **cyclic algebra** over a cyclic operad $\mathcal{C}$ is a $\mathcal{C}$ -algebra structure on a vector space V together with a nondegenerate invariant bilinear form $B : V \otimes V \rightarrow \mathbf{k}$.*

By [17], a cyclic algebra is the same as a cyclic operad homomorphism $\mathcal{C} \rightarrow \mathcal{E}nd_V$, where $\mathcal{E}nd_V$ is the cyclic endomorphism operad of the pair (V, B) recalled in Example 8.1.

EXAMPLE 8.4 *A cyclic algebra over the cyclic operad $\mathcal{C}om$ is a **Frobenius algebra**, that is, a structure consisting of a commutative associative multiplication $\cdot : V \otimes V \rightarrow V$ as in Example 5.1 together with a non-degenerate symmetric bilinear form $B : V \otimes V \rightarrow \mathbf{k}$, invariant in the sense that*

$$B(a \cdot b, c) = B(a, b \cdot c), \quad \text{for all } a, b, c \in V.$$

Similarly, a cyclic Lie algebra is given by a Lie bracket $[-, -] : V \otimes V \rightarrow V$ and a non-degenerate symmetric bilinear form $B : V \otimes V \rightarrow \mathbf{k}$ satisfying

$$B([a, b], c) = B(a, [b, c]), \quad \text{for } a, b, c \in V.$$

For algebras over cyclic operads, one may introduce cyclic cohomology that generalizes the classical cyclic cohomology of associative algebras [64–66] as the non-abelian derived functor of the universal bilinear form [9], [17]. Let us close this Section by mentioning two examples of operads with other types of higher symmetries. The symmetry required for *anticyclic operads* differs from the symmetry of cyclic operads by the sign [17]. *Dihedral operads* exhibit a symmetry governed by the dihedral groups [63].

9 Modular Operads

Let us consider again the Σ^+ -module $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ of Riemann spheres with punctures. We saw that the operation $M, N \mapsto M \circ_i N$ of sewing the 0th

hole of the surface N to the i th hole of the surface M defined on $\widehat{\mathfrak{M}}_0$ a cyclic operad structure. One may generalize this operation by defining, for $M \in \widehat{\mathfrak{M}}_0(m)$, $N \in \widehat{\mathfrak{M}}_0(n)$, $0 \leq i \leq m$, $0 \leq j \leq n$, the element $M_{i \circ_j} N \in \widehat{\mathfrak{M}}_0(m+n-1)$ by sewing the j th hole of M to the i th hole of N . Under this notation, $\circ_i = i \circ_0$. In the same manner, one may consider a single surface $M \in \widehat{\mathfrak{M}}_0(n)$, choose labels i, j , $0 \leq i \neq j \leq n$, and sew the i th hole of M along the j th hole of the *same* surface. The result is a new surface $\xi_{\{i,j\}}(M)$, with $n-2$ holes and genus 1.

This leads us to the system $\widehat{\mathfrak{M}} = \{\widehat{\mathfrak{M}}(g, n)\}_{g \geq 0, n \geq -1}$, where $\widehat{\mathfrak{M}}(g, n)$ denotes now the moduli space of genus g Riemann surfaces with $n+1$ holes. Observe that we include $\widehat{\mathfrak{M}}(g, n)$ also for $n = -1$; $\widehat{\mathfrak{M}}(g, -1)$ is the moduli space of Riemann surfaces of genus g . The operations $i \circ_j$ and $\xi_{\{i,j\}}$ act on $\widehat{\mathfrak{M}}$. Clearly, for $M \in \widehat{\mathfrak{M}}(g, m)$ and $N \in \widehat{\mathfrak{M}}(h, n)$, $0 \leq i \leq m$, $0 \leq j \leq n$ and $g, h \geq 0$,

$$M_{i \circ_j} N \in \widehat{\mathfrak{M}}(g+h, m+n-1) \quad (9.1)$$

and, for $m \geq 1$ and $g \geq 0$,

$$\xi_{\{i,j\}}(M) \in \widehat{\mathfrak{M}}(g+1, m-2). \quad (9.2)$$

A particular case of (9.1) is the non-operadic composition

$$_0 \circ_0 : \widehat{\mathfrak{M}}(g, 0) \times \widehat{\mathfrak{M}}(h, 0) \rightarrow \widehat{\mathfrak{M}}(g+h, -1), \quad g, h \geq 0. \quad (9.3)$$

Modular operads are abstractions of the above structure satisfying a certain additional stability condition. The following definitions, taken from [67], are made for the category of $\mathbf{k}$ -modules, but they can be easily generalized to an arbitrary symmetric monoidal category with finite colimits, whose monoidal product $\odot$ is distributive over colimits. Let us introduce the underlying category for modular operads.

A *modular Σ -module* is a sequence $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ of $\mathbf{k}$ -modules such that each $\mathcal{E}(g, n)$ has a right $\mathbf{k}[\Sigma_n^+]$ -action. We say that $\mathcal{E}$ is *stable* if

$$\mathcal{E}(g, n) = 0 \quad \text{for } 2g + n - 1 \leq 0 \quad (9.4)$$

and denote $\mathbf{MMod}$ the category of stable modular Σ -modules.

Stability (9.4) says that $\mathcal{E}(g, n)$ is trivial for $(g, n) = (0, -1)$, $(1, -1)$, $(0, 0)$ and $(0, 1)$. We will sometimes express the stability of $\mathcal{E}$ by writing $\mathcal{E} = \{\mathcal{E}(g, n)\}_{(g, n) \in \mathfrak{S}}$, where

$$\mathfrak{S} := \{(g, n) \mid g \geq 0, n \geq -1 \text{ and } 2g + n - 1 > 0\}.$$

Recall that a genus g Riemann surface with k marked points is stable if it does not admit infinitesimal automorphisms. This happens if and only if $2(g - 1) + k > 0$, that is, excluded is the torus with no marked points and the sphere with less than three marked points. Thus the stability property of modular Σ -modules is analogous to the stability of Riemann surfaces.

Now we introduce graphs that serve as pasting schemes for modular operads. The naive notion of a graph as we have used it up to this point is not subtle enough; we need to replace it by a more sophisticated:

DEFINITION 9.1 *A **graph** Γ is a finite set $\text{Flag}(\Gamma)$ (whose elements are called **flags** or **half-edges**) together with an involution σ and a partition λ . The **vertices** $\text{vert}(\Gamma)$ of a graph Γ are the blocks of the partition λ , we assume also that the number of these blocks is finite. The **edges** $\text{Edg}(\Gamma)$ are pairs of flags forming a two-cycle of σ . The **legs** $\text{Leg}(\Gamma)$ are the fixed points of σ .*

We also denote by $\text{edge}(v)$ the flags belonging to the block v or, in common speech, half-edges adjacent to the vertex v . We say that graphs Γ_1 and Γ_2 are *isomorphic* if there exists a set isomorphism $\varphi : \text{Flag}(\Gamma_1) \rightarrow \text{Flag}(\Gamma_2)$ that preserves the partitions and commutes with the involutions. We may associate to a graph Γ a finite one-dimensional cell complex $|\Gamma|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag, a point for each block of the partition, and imposing the following equivalence relation: The points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ with the point corresponding to the block, and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ .

We call $|\Gamma|$ the *geometric realization* of Γ . Observe that empty blocks of the partition generate isolated vertices in the geometric realization. We will usually make no distinction between the graph and its geometric realization. As an example (taken from [67]), consider the graph with $\{a, b, \dots, i\}$ as the set of flags, the involution $\sigma = (df)(eg)$ and the partition $\{a, b, c, d, e\} \cup \{f, g, h, i\}$. The geometric realization of this graph is the ‘sputnik’ in Fig. 18. Let us introduce labeled

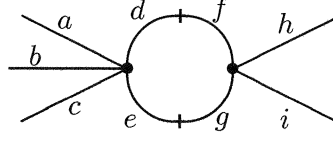


Figure 18: The sputnik.

versions of the above notions. A *(vertex-) labeled graph* is a connected graph Γ together with a map g (the *genus map*) from $\text{vert}(\Gamma)$ to the set $\{0, 1, 2, \dots\}$. Labeled graphs Γ_1 and Γ_2 are isomorphic if there exists an isomorphism preserving the labels of the vertices. The *genus* $g(\Gamma)$ of a labeled graph Γ is defined by

$$g(\Gamma) := b_1(\Gamma) + \sum_{v \in \text{vert}(\Gamma)} g(v), \quad (9.5)$$

where $b_1(\Gamma) := \dim H_1(|\Gamma|)$ is the first Betti number of the graph $|\Gamma|$, i.e. the number of independent circuits of Γ . A graph Γ is *stable* if

$$2(g(v) - 1) + |\text{edge}(v)| > 0,$$

at each vertex $v \in \text{vert}(\Gamma)$.

For $g \geq 0$ and $n \geq -1$, let $\mathbf{MGr}(g, n)$ be the groupoid whose objects are pairs (Γ, ℓ) consisting of a stable (vertex-) labeled graph Γ of genus g and an isomorphism $\ell : \text{Leg}(\Gamma) \rightarrow \{0, \dots, n\}$ labeling the legs of Γ by elements of $\{0, \dots, n\}$. Morphisms of $\mathbf{MGr}(g, S)$ are isomorphisms of vertex-labeled graphs preserving the labeling of the legs. The stability implies, via an elementary combinatorial topology that, for each fixed $g \geq 0$ and $n \geq -1$, there is only a finite number of isomorphism classes of stable graphs $\Gamma \in \mathbf{MGr}(g, n)$, see [67].

We will also need the following obvious generalization of (8.1): if $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ is a modular Σ -module and X a set with $n + 1$ elements, then

$$\mathcal{E}((g, X)) := \mathcal{E}(g, n) \times_{\Sigma_n^+} \text{Bij}([n]^+, X), \quad g \geq 0, \quad n \geq -1. \quad (9.6)$$

For a modular Σ -module $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ and a labeled graph Γ , let $\mathcal{E}(\Gamma)$ be the product

$$\mathcal{E}(\Gamma) := \bigotimes_{v \in \text{vert}(\Gamma)} \mathcal{E}((g(v), \text{edge}(v))). \quad (9.7)$$

Evidently, the correspondence $\Gamma \mapsto \mathcal{E}(\Gamma)$ defines a functor from the category $\mathbf{MGr}(g, n)$ to the category of $\mathbf{k}$ -modules and their isomorphisms. We may thus define an endofunctor $\mathbb{M}$ on the category $\mathbf{Mod}$ of stable modular Σ -modules by the formula

$$\mathbb{M}\mathcal{E}(g, n) := \operatorname{colim}_{\Gamma \in \mathbf{MGr}(g, n)} \mathcal{E}(\Gamma), \quad g \geq 0, \quad n \geq -1.$$

Choosing a representative for each isomorphism class in $\mathbf{MGr}(g, n)$, one obtains the identification

$$\mathbb{M}\mathcal{E}(g, n) \cong \bigoplus_{[\Gamma] \in \{\mathbf{MGr}(g, n)\}} \mathcal{E}(\Gamma)_{\operatorname{Aut}(\Gamma)}, \quad g \geq 0, \quad n \geq -1, \quad (9.8)$$

where $\{\mathbf{MGr}(g, n)\}$ is the set of isomorphism classes of objects of the groupoid $\mathbf{MGr}(g, n)$ and the subscript $\operatorname{Aut}(\Gamma)$ denotes the space of coinvariants. Stability (9.4) implies that the summation in the right-hand side of (9.8) is finite. Formula (9.8) generalizes (6.10) which does not contain coinvariants because there are no nontrivial automorphisms of leaf-labeled trees. On the other hand, stable labeled graphs with nontrivial automorphisms are abundant, an example can be easily constructed from the graph in Figure 18. The functor $\mathbb{M}$ carries a triple structure of ‘erasing the braces’ similar to the one used on pages 62 and 65.

DEFINITION 9.2 *A **modular operad** is an algebra over the triple $\mathbb{M} : \mathbf{Mod} \rightarrow \mathbf{Mod}$.*

Therefore a modular operad is a stable modular Σ -module $\mathcal{A} = \{\mathcal{A}(g, n)\}_{(g, n) \in \mathbb{S}}$ equipped with operations that determine coherent contractions along stable modular graphs. Observe that the stability condition is built firmly into the very definition. Very crucially, modular operads *do not have units*, because such a unit ought to be an element of the space $\mathcal{A}(0, 1)$ which is empty, by (9.4).

One can easily introduce un-stable modular operads and their unital versions, but the main motivating example reviewed below is stable. We will consider an extension of the Grothendieck-Knudsen configuration operad $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ consisting of moduli spaces of stable curves of arbitrary genera in the sense of the following generalization of Definition 4.1:

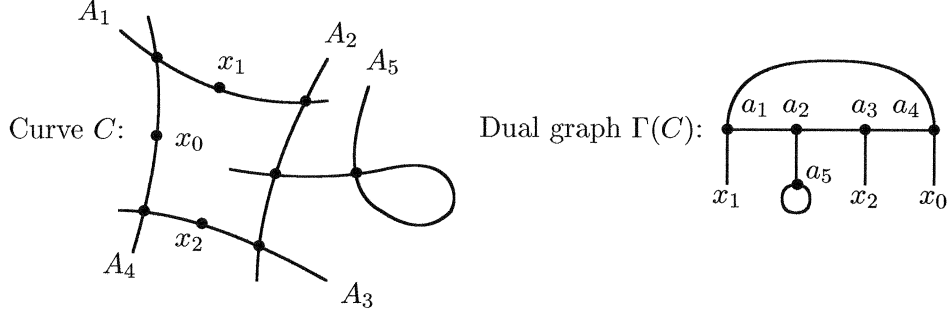


Figure 19: A stable curve and its dual graph. The curve C on the left has five components A_i , $1 \leq i \leq 5$, and three marked points x_0 , x_1 and x_2 . The dual graph $\Gamma(C)$ on the right has five vertices a_i , $1 \leq i \leq 5$, corresponding to the components of the curve and three legs labeled by the marked points.

DEFINITION 9.3 A **stable $(n+1)$ -pointed curve**, $n \geq 0$, is a connected complex projective curve C with at most nodal singularities, together with a ‘marking’ given by a choice $x_0, \dots, x_n \in C$ of smooth points. The stability means, as usual, that there are no infinitesimal automorphisms of C fixing the marked points and double points.

The stability in Definition 9.3 is equivalent to saying that each smooth component of C isomorphic to the complex projective space $\mathbb{CP}^1$ has at least three special points and that each smooth component isomorphic to the torus has at least one special point, where by a special point we mean either a double point or a node.

The *dual graph* $\Gamma = \Gamma(C)$ of a stable $(n+1)$ -pointed curve $C = (C, x_0, \dots, x_n)$ is a labeled graph whose vertices are the components of C , edges are the nodes and its legs are the points $\{x_i\}_{0 \leq i \leq n}$. An edge e_y corresponding to a nodal point y joins the vertices corresponding to the components intersecting at y . The vertex v_K corresponding to a branch K is labeled by the genus of the normalization of K . See [31] for the normalization and recall that a curve is normal if and only if it is nonsingular. The construction of $\Gamma(C)$ from a curve C is visualized in Figure 19.

Let us denote by $\overline{\mathcal{M}}_{g,n+1}$ the coarse moduli space [31] of stable $(n+1)$ -pointed curves C such that the dual graph $\Gamma(C)$ has genus g , in the sense of (9.5). The

genus of $\Gamma(C)$ in fact equals the arithmetic genus of the curve C , thus $\overline{\mathcal{M}}_{g,n+1}$ is the coarse moduli space of stable curves of arithmetic genus g with $n+1$ marked points. By a result of P. Deligne, F.F. Knudsen and D. Mumford [22, 32, 33], $\overline{\mathcal{M}}_{g,n+1}$ is a projective variety.

Observe that, for a curve $C \in \overline{\mathcal{M}}_{0,n+1}$, the graph $\Gamma(C)$ must necessarily be a tree and all components of C must be smooth of genus 0, therefore $\overline{\mathcal{M}}_{0,n+1}$ coincides with the moduli space $\overline{\mathcal{M}}_0(n)$ of genus 0 stable curves with $n+1$ marked points that we discussed in Section 4. Dual graphs of curves $C \in \overline{\mathcal{M}}_{g,n+1}$ are stable labeled graphs belonging to $\mathbf{MGr}(g, n+1)$.

The symmetric group Σ_n^+ acts on $\overline{\mathcal{M}}_{g,n+1}$ by renumbering the marked points, therefore

$$\overline{\mathcal{M}} := \{\overline{\mathcal{M}}(g, n)\}_{g \geq 0, n \geq -1},$$

with $\overline{\mathcal{M}}(g, n) := \overline{\mathcal{M}}_{g,n+1}$, is a modular Σ -module in the category of projective varieties. Since there are no stable curves of genus g with $n+1$ punctures if $2g+n-1 \leq 0$, $\overline{\mathcal{M}}$ is a *stable* modular Σ -module. Let us define the contraction along a stable graph $\Gamma \in \mathbf{MGr}(g, n)$

$$\alpha_\Gamma : \overline{\mathcal{M}}(\Gamma) = \prod_{v \in \text{vert}(\Gamma)} \overline{\mathcal{M}}(g(v), \text{edge}(v)) \rightarrow \overline{\mathcal{M}}(g, n) \quad (9.9)$$

by gluing the marked points of curves from $\overline{\mathcal{M}}(g(v), \text{edge}(v))$, $v \in \text{vert}(\Gamma)$, according to the graph Γ . To be more precise, let

$$\prod_{v \in \text{vert}(\Gamma)} C_v, \quad \text{where } C_v \in \overline{\mathcal{M}}(g(v), \text{edge}(v)),$$

be an element of $\overline{\mathcal{M}}(\Gamma)$. Let e be an edge of the graph Γ connecting vertices v_1 and v_2 , $e = \{y_{v_1}^e, y_{v_2}^e\}$, where $y_{v_i}^e$ is a marked point of the component C_{v_i} , $i = 1, 2$, which is also the name of the corresponding flag of the graph Γ . The curve $\alpha_\Gamma(C)$ is then obtained by the identifications $y_{v_1}^e = y_{v_2}^e$, introducing a nodal singularity, for all $e \in \text{Edg}(\Gamma)$. The procedure is the same as that described for the tree level in Section 4. As proved in [67, § 6.2], the contraction maps (9.9) define on the stable modular Σ -module of coarse moduli spaces $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(g, n)\}_{(g,n) \in \mathbb{S}}$ a modular operad structure in the category of complex projective varieties.

Let us look more closely at the structure of the modular triple $\mathbb{M}$. Given a (stable or unstable) modular Σ -module $\mathcal{E}$, there is, for each $g \geq 0$ and $n \geq -1$, a

natural decomposition

$$\mathbb{M}(\mathcal{E})(g, n) = \mathbb{M}_0(\mathcal{E})(g, n) \oplus \mathbb{M}_1(\mathcal{E})(g, n) \oplus \mathbb{M}_2(\mathcal{E})(g, n) \oplus \cdots,$$

with $\mathbb{M}_k(\mathcal{E})(g, n)$ the subspace obtained by summing over graphs Γ with $\dim H_1(|\Gamma|) = k$, $k \geq 0$. In particular, $\mathbb{M}_0(\mathcal{E})(g, n)$ is a summation over simply connected graphs. It is not difficult to see that $\mathbb{M}_0(\mathcal{E})$ is a *subtriple* of $\mathbb{M}(\mathcal{E})$. This shows that modular operads are $\mathbb{M}_0$ -algebras with some additional operations (the ‘contractions’) that raise the genus and generate the higher components $\mathbb{M}_k$, $k \geq 1$, of the modular triple $\mathbb{M}$.

There seems to be a belief expressed in the proof of Theorem in [67] that, in the stable case, the triple $\mathbb{M}_0$ is equivalent to the non-unital cyclic operad triple Ψ_+ , but it is not so. The triple $\mathbb{M}_0$ is *much bigger*, for example, if $a \in \mathcal{E}(1, 0)$, then $\mathbb{M}_0(\mathcal{E})(2, -1)$ contains a non-operadic element

$$\begin{array}{c} a \qquad \qquad \qquad a \\ \bullet \text{-----} \bullet \end{array}$$

which can be also written, using (9.3), as $a_0 \circ_0 a$. The corresponding part $\Psi_+(\mathcal{E})(-1)$ of the cyclic triple is empty. In the Grothendieck-Knudsen modular operad $\overline{\mathcal{M}}$, an element of the above type is realized by two tori meeting at a nodal point.

On the other hand, the triple $\mathbb{M}_0$ restricted to the subcategory of stable modular Σ -modules $\mathcal{E}$ such that $\mathcal{E}(g, n) = 0$ for $g > 0$ indeed coincides with the non-unital cyclic operad triple Ψ_+ , as was in fact proved in [67]. Therefore, given a modular operad $\mathcal{A} = \{\mathcal{A}(g, n)\}_{(g, n) \in \mathfrak{G}}$, there is an induced non-unital cyclic operad structure on the cyclic collection $\mathcal{A}^b := \{\mathcal{A}(0, n)\}_{n \geq 2}$. We will call $\mathcal{A}^b$ the *associated cyclic operad*. For example, the cyclic operad associated to the Grothendieck-Knudsen modular operad $\overline{\mathcal{M}}$ equals its genus zero part $\overline{\mathcal{M}}_0$.

A *biased* definition of modular operads can be found in [17]. It is formulated in terms of operations

$$\{i \circ_j : \mathcal{A}(g, m) \otimes \mathcal{A}(h, n) \rightarrow \mathcal{A}(g + h, m + n); \ 0 \leq i \leq m, \ 0 \leq j \leq n, \ g, h \geq 0\}$$

together with contractions

$$\{\xi_{\{i, j\}} : \mathcal{A}(g, m) \rightarrow \mathcal{A}(g + 1, m - 2); \ m \geq 1, \ g \geq 0\}$$

that generalize (9.1) and (9.2).

EXAMPLE 9.1 Let $V = (V, B)$ be a vector space with a symmetric inner product $B : V \otimes V \rightarrow \mathbf{k}$. Denote, for each $g \geq 0$ and $n \geq -1$,

$$\mathcal{E}nd_V(g, n) := V^{\otimes(n+1)}.$$

It is clear from definition (9.7) that, for any labeled graph $\Gamma \in \mathbf{MGr}(g, n)$, $\mathcal{E}nd_V(\Gamma) = V^{\otimes \text{Flag}(\Gamma)}$.

Let $B^{\otimes \text{Edg}(\Gamma)} : V^{\otimes \text{Flag}(\Gamma)} \rightarrow V^{\otimes \text{Leg}(\Gamma)}$ be the multilinear form which contracts the factors of $V^{\otimes \text{Flag}(\Gamma)}$ corresponding to the flags which are paired up as edges of Γ . Then we define $\alpha_\Gamma : \mathcal{E}nd_V(\Gamma) \rightarrow \mathcal{E}nd_V(g, n)$ to be the map

$$\alpha_\Gamma : \mathcal{E}nd_V(\Gamma) = V^{\otimes \text{Flag}(\Gamma)} \xrightarrow{B^{\otimes \text{Edg}(\Gamma)}} V^{\otimes \text{Leg}(\Gamma)} \xrightarrow{V^{\otimes \ell}} V^{\otimes(n+1)} = \mathcal{E}nd_V(g, n),$$

where $\ell : \text{Leg}(\Gamma) \rightarrow \{0, \dots, n\}$ is the labeling of the legs of Γ . It is easy to show that the compositions $\{\alpha_\Gamma; \Gamma \in \mathbf{MGr}(g, n)\}$ define on $\mathcal{E}nd_V$ the structure of an un-stable unital modular operad, see [67].

An algebra over a modular operad $\mathcal{A}$ is a vector space V with an inner product B , together with a morphism $\rho : \mathcal{A} \rightarrow \mathcal{E}nd_V$ of modular operads. Several important structures are algebras over modular operads. For example, an algebra over the homology $H_*(\overline{\mathcal{M}})$ of the Grothendieck-Knudsen modular operad is the same as a cohomological field theory in the sense of [10]. Other physically relevant algebras over modular operads can be found in [17, 34, 67]. Relations between modular operads, chord diagrams and Vassiliev invariants are studied in [62].

10 PROPS

Operads are devices invented to describe structures consisting of operations with several inputs and *one* output. There are, however, important structures with operations having several inputs and *several* outputs. Let us recall the most prominent one:

EXAMPLE 10.1 A (associative) **bialgebra** is a $\mathbf{k}$ -module V with a **multiplication** $\mu : V \otimes V \rightarrow V$ and a **comultiplication** (also called a **diagonal**) $\Delta : V \rightarrow V \otimes V$. The multiplication is associative:

$$\mu(\mu \otimes id_V) = \mu(id_V \otimes \mu),$$

the comultiplication is coassociative:

$$(\Delta \otimes \text{id}_V)\Delta = (\text{id}_V \otimes \Delta)\Delta$$

and the usual compatibility between μ and Δ is assumed:

$$\Delta(u \cdot v) = \Delta(u) \cdot \Delta(v) \quad \text{for } u, v \in V, \quad (10.1)$$

where $u \cdot v := \mu(u, v)$ and the dot $\cdot$ in the right hand side denotes the multiplication induced on $V \otimes V$ by μ . Loosely speaking, bialgebras are Hopf algebras without unit, counit and antipode.

PROPs (an abbreviation of **product** and **permutation category**) describe structures as in Example 10.1. Although PROPs are more general than operads, they appeared much sooner, in a 1965 Mac Lane's paper [35]. This might be explained by the fact that the definition of PROPs is more compact than that of operads – compare Definition 10.1 below with Definition 1.1 in Section 1 and Definition 3.1 in Section 3. PROPs then entered the ‘renaissance of operads’ in 1996 via [36].

Definition 10.1 uses the notion of a symmetric strict monoidal category, see [17, 35, 61]. An example is the category $\mathbf{Mod}_{\mathbf{k}}$ of $\mathbf{k}$ -modules, with the monoidal product $\odot$ given by the tensor product $\otimes = \otimes_{\mathbf{k}}$, the symmetry $S_{U,V} : U \otimes V \rightarrow V \otimes U$ defined as $S_{U,V}(u, v) := v \otimes u$ for $u \in U$ and $v \in V$, and the unit 1 the ground ring $\mathbf{k}$.

DEFINITION 10.1 *A ($\mathbf{k}$ -linear) PROP (called a theory in [36]) is a symmetric strict monoidal category $\mathbf{P} = (\mathbf{P}, \odot, S, 1)$ enriched over $\mathbf{Mod}_{\mathbf{k}}$ such that*

- (i) *the objects are indexed by (or identified with) the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, and*
- (ii) *the product satisfies $m \odot n = m + n$, for any $m, n \in \mathbb{N} = \text{Ob}(\mathbf{P})$ (hence the unit 1 equals 0).*

Recall that the $\mathbf{Mod}_{\mathbf{k}}$ -enrichment in the above definition means that each hom-set $\text{Mor}_{\mathbf{P}}(m, n)$ is a $\mathbf{k}$ -module and the operations of the monoidal category $\mathbf{P}$ (the composition $\circ$, the product $\odot$ and the symmetry S) are compatible with this $\mathbf{k}$ -linear structure.

For a PROP $\mathbf{P}$ denote $\mathbf{P}(m, n) := \text{Mor}_{\mathbf{P}}(m, n)$. The symmetry S induces, via the canonical identifications $m \cong \mathbf{1}^{\odot m}$ and $n \cong \mathbf{1}^{\odot n}$, on each $\mathbf{P}(m, n)$ a structure of (Σ_m, Σ_n) -bimodule (left Σ_m - right Σ_n -module such that the left action commutes with the right one). Therefore a PROP is a collection $\mathbf{P} = \{\mathbf{P}(m, n)\}_{m, n \geq 0}$ of (Σ_m, Σ_n) -bimodules, together with two types of compositions, *horizontal*

$$\otimes : \mathbf{P}(m_1, n_1) \otimes \cdots \otimes \mathbf{P}(m_s, n_s) \rightarrow \mathbf{P}(m_1 + \cdots + m_s, n_1 + \cdots + n_s),$$

induced, for all $m_1, \dots, m_s, n_1, \dots, n_s \geq 0$, by the monoidal product $\odot$ of $\mathbf{P}$, and *vertical*

$$\circ : \mathbf{P}(m, n) \otimes \mathbf{P}(n, k) \rightarrow \mathbf{P}(m, k),$$

given, for all $m, n, k \geq 0$, by the categorial composition. The monoidal unit is an element $e := \mathbf{1} \in \mathbf{P}(1, 1)$. In Definition 10.1, $\mathbf{Mod}_{\mathbf{k}}$ can be replaced by an arbitrary symmetric strict monoidal category.

Let $\mathbf{P} = \{\mathbf{P}(m, n)\}_{m, n \geq 0}$ and $\mathbf{Q} = \{\mathbf{Q}(m, n)\}_{m, n \geq 0}$ be two PROPs. A *homomorphism* $f : \mathbf{P} \rightarrow \mathbf{Q}$ is a sequence $f = \{f(m, n) : \mathbf{P}(m, n) \rightarrow \mathbf{Q}(m, n)\}_{m, n \geq 0}$ of bi-equivariant maps which commute with both the vertical and horizontal compositions. An *ideal* in a PROP $\mathbf{P}$ is a system $\mathbf{I} = \{\mathbf{I}(m, n)\}_{m, n \geq 0}$ of left Σ_m - right Σ_n -invariant subspaces $\mathbf{I}(m, n) \subset \mathbf{P}(m, n)$ which is closed, in the obvious sense, under both the vertical and horizontal compositions. Kernels, images, etc., of homomorphisms between PROPs, as well as quotients of PROPs by PROPic ideals, are defined componentwise, see [36–39] for details.

EXAMPLE 10.2 *The endomorphism PROP of a $\mathbf{k}$ -module V is the system*

$$\mathcal{E}nd_V = \{\mathcal{E}nd_V(m, n)\}_{m, n \geq 0}$$

with $\mathcal{E}nd_V(m, n)$ the space of linear maps $\text{Lin}(V^{\otimes n}, V^{\otimes m})$ with n ‘inputs’ and m ‘outputs,’ $e \in \mathcal{E}nd_V(1, 1)$ the identity map, horizontal composition given by the tensor product of linear maps, and vertical composition by the ordinary composition of linear maps.

Also algebras over PROPs can be introduced in a very concise way:

DEFINITION 10.2 *A **P-algebra** is a strict symmetric monoidal functor $\lambda : \mathbf{P} \rightarrow \mathbf{Mod}_{\mathbf{k}}$ of enriched monoidal categories. The value $\lambda(1)$ is the underlying space of the algebra ρ .*

It is easy to see that a $\mathbf{P}$ -algebra is the same as a PROP homomorphism $\rho : \mathbf{P} \rightarrow \mathcal{E}nd_V$. As in Proposition 5.1, a $\mathbf{P}$ -algebra is determined by a system

$$\alpha : \mathbf{P}(m, n) \otimes V^{\otimes n} \rightarrow V^{\otimes m}, \quad m, n, \geq 0,$$

of linear maps satisfying appropriate axioms.

As before, the first step in formulating an unbiased definition of PROPs is to specify their underlying category. A Σ -bimodule is a system $E = \{E(m, n)\}_{m, n \geq 0}$ such that each $E(m, n)$ is a left $\mathbf{k}[\Sigma_m]$ - right $\mathbf{k}[\Sigma_n]$ -bimodule. Let $\Sigma\text{-bimod}$ denote the category of Σ -bimodules. For $E \in \Sigma\text{-bimod}$ and finite sets Y, X with m resp. n elements put

$$E(Y, X) := \text{Bij}(Y, [m]) \times_{\Sigma_m} E(m, n) \times_{\Sigma_n} \text{Bij}([n], X), \quad m, n \geq 0,$$

where $\text{Bij}(-, -)$ is the same as in (6.6). Pasting schemes for PROPs are *directed* (m, n) -graphs, by which we mean finite, not necessary connected, graphs in the sense of Definition 9.1 such that

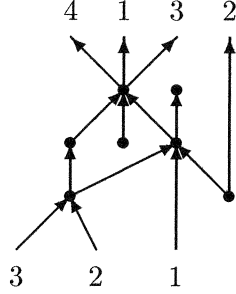
- (i) each edge is equipped with a direction
- (ii) there are no directed cycles and
- (iii) the set of legs is divided into the set of inputs labeled by $\{1, \dots, n\}$ and the set of outputs labeled by $\{1, \dots, m\}$.

An example of a directed graph is given in Figure 20. We denote by $\mathbf{Gr}(m, n)$ the category of directed (m, n) -graphs and their isomorphisms. The direction of edges determines at each vertex $v \in \text{vert}(G)$ of a directed graph G a disjoint decomposition

$$\text{edge}(v) = \text{in}(v) \sqcup \text{out}(v)$$

of the set of edges adjacent to v into the set $\text{in}(v)$ of incoming edges and the set $\text{out}(v)$ of outgoing edges. The pair $(\#(\text{out}(v)), \#(\text{in}(v))) \in \mathbb{N} \times \mathbb{N}$ is called the *biarity* of v . To incorporate the unit, we need to extend the category $\mathbf{Gr}(m, n)$, for $m = n$, into the category $\mathbf{UGr}(m, n)$ by allowing the exceptional graph

$$\uparrow \uparrow \uparrow \cdots \uparrow \in \mathbf{UGr}(n, n), \quad n \geq 1,$$

Figure 20: A directed graph from $\mathbf{Gr}(4, 3)$.

with n inputs, n outputs and no vertices. For a graph $G \in \mathbf{UGr}(m, n)$ and a Σ -bimodule E , let

$$E(G) := \bigotimes_{v \in \text{vert}(G)} E(\text{out}(v), \text{in}(v)).$$

and

$$\Gamma_{\mathbf{P}}(E)(m, n) := \text{colim}_{G \in \mathbf{UGr}(m, n)} E(G), \quad m, n \geq 0. \quad (10.2)$$

The Σ -bimodule $\Gamma_{\mathbf{P}}(E)$ is a PROP, with the vertical composition given by the disjoint union of graphs, the horizontal composition by grafting the legs, and the unit the exceptional graph $\uparrow \in \Gamma_{\mathbf{P}}(E)(1, 1)$. The following proposition follows from [40] and [37–39]:

PROPOSITION 10.1 *The PROP $\Gamma_{\mathbf{P}}(E)$ is the free PROP generated by the Σ -bimodule E .*

As in the previous Sections, (10.2) defines a triple $\Gamma_{\mathbf{P}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ with the triple multiplication of erasing the braces. According to general principles [30], Proposition 10.1 is almost equivalent to

PROPOSITION 10.2 *PROPs are algebras over the triple $\Gamma_{\mathbf{P}}$.*

One may obviously consider *non-unital PROPs* defined as algebras over the triple

$$\Psi_{\mathbf{P}}(E)(m, n) := \text{colim}_{G \in \mathbf{Gr}(m, n)} E(G), \quad m, n \geq 0,$$

and develop a theory parallel to the theory of non-unital operads reviewed in Section 4.

EXAMPLE 10.3 *We will use the graphical language explained in Example 6.3. Let $\Gamma(\wedge, \Upsilon)$ be the free PROP generated by one operation $\wedge$ of biarity $(1, 2)$ and one operation Υ of biarity $(2, 1)$. As we noticed already in [36, 41], the PROP $\mathbf{B}$ describing bialgebras equals*

$$\mathbf{B} = \Gamma(\wedge, \Upsilon) / \mathbf{l}_{\mathbf{B}},$$

where $\mathbf{l}_{\mathbf{B}}$ is the PROPic ideal generated by

$$\wedge - \wedge, \quad \Upsilon - \Upsilon \quad \text{and} \quad \chi - \text{diagram}.$$
(10.3)

In the above display we denoted

$$\begin{aligned} \wedge &:= \wedge \circ (\wedge \otimes e), \quad \wedge := \wedge \circ (e \otimes \wedge), \quad \Upsilon := (\Upsilon \otimes e) \circ \Upsilon, \quad \Upsilon := (e \otimes \Upsilon) \circ \Upsilon, \\ \chi &:= \Upsilon \circ \wedge \quad \text{and} \quad \text{diagram} := (\wedge \otimes \wedge) \circ \kappa \circ (\Upsilon \otimes \Upsilon), \end{aligned}$$

where $\kappa \in \Sigma_4$ is the permutation

$$\kappa := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ | \quad \times \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}.$$
(10.4)

The above description of $\mathbf{B}$ is ‘tautological,’ but B. Enriquez and P. Etingof found in [?] the following basis of the $\mathbf{k}$ -linear space $\mathbf{B}(m, n)$ for arbitrary $m, n \geq 1$. Let $\wedge \in \mathbf{B}(1, 2)$ be the equivalence class, in $\mathbf{B} = \Gamma(\wedge, \Upsilon) / \mathbf{l}_{\mathbf{B}}$, of the generator $\wedge \in \Gamma(\wedge, \Upsilon)(1, 2)$ (we use the same symbol both for a generator and its equivalence class). Define $\wedge^{[1]} := e \in \mathbf{B}(1, 1)$ and, for $a \geq 2$, let

$$\wedge^{[a]} := \wedge \circ (\wedge \otimes e) \circ (\wedge \otimes e^{\otimes 2}) \circ \dots \circ (\wedge \otimes e^{\otimes (a-2)}) \in \mathbf{B}(1, a).$$

Let $\Upsilon_{[b]} \in \mathbf{B}(b, 1)$ has the obvious similar meaning. The elements

$$(\wedge^{[a_1]} \otimes \dots \otimes \wedge^{[a_m]}) \circ \sigma \circ (\Upsilon_{[b^1]} \otimes \dots \otimes \Upsilon_{[b^n]}),$$
(10.5)

where $\sigma \in \Sigma_N$ for some $N \geq 1$, and $a_1 + \dots + a_m = b^1 + \dots + b^n = N$, form a $\mathbf{k}$ -linear basis of $\mathbf{B}(m, n)$. This result can also be found in [42]. See also [43, 44] for the bialgebra PROP viewed from a different perspective.

EXAMPLE 10.4 *Each operad $\mathcal{P}$ generates a unique PROP $\mathbf{P}$ such that $\mathbf{P}(1, n) = \mathcal{P}(n)$ for each $n \geq 0$. The components of such a PROP are given by*

$$\mathbf{P}(m, n) = \bigoplus_{r_1 + \dots + r_k = n} [\mathcal{P}(1, r_1) \otimes \dots \otimes \mathcal{P}(1, r_k)] \times_{\Sigma_{r_1} \times \dots \times \Sigma_{r_k}} \Sigma_n,$$

for each $m, n \geq 0$. The (topological) PROPs considered in [24] are all of this type. On the other hand, Example 10.3 shows that not each PROP is of this form. A PROP $\mathbf{P}$ is generated by an operad if and only if it has a presentation $\mathbf{P} = \Gamma_{\mathbf{P}}(E)/(R)$, where E is a Σ -bimodule such that $E(m, n) = 0$ for $m \neq 1$ and R is generated by elements in $\Gamma_{\mathbf{P}}(E)(1, n)$, $n \geq 0$.

11 Properads, Dioperads and $\frac{1}{2}$ PROPs

As we saw in Proposition 6.1, under some mild assumptions, the components of free operads are finite-dimensional. In contrast, PROPs are huge objects. For example, the component $\Gamma_{\mathbf{P}}(\wedge, \vee)(m, n)$ of the free PROP $\Gamma_{\mathbf{P}}(\wedge, \vee)$ used in the definition of the bialgebra PROP $\mathbf{B}$ in Example 10.3 is infinite-dimensional for each $m, n \geq 1$, and also the components of the bialgebra PROP $\mathbf{B}$ itself are infinite-dimensional, as follows from the fact that the Enriquez-Etingof basis (10.5) of $\mathbf{B}(m, n)$ has, for $m, n \geq 1$, infinitely many elements.

To handle this combinatorial explosion of PROPs combined with lack of suitable filtrations, smaller versions of PROPs were invented. Let us begin with the simplest modification which we use as an example which explains the general scheme of modifying PROPs. Denote $\mathbf{UGr}_c(m, n)$ the full subcategory of $\mathbf{UGr}(m, n)$ consisting of *connected* graphs and consider the triple defined by

$$\Gamma_c(E)(m, n) := \operatorname{colim}_{G \in \mathbf{UGr}_c(m, n)} E(G), \quad m, n \geq 0, \quad (11.1)$$

for $E \in \Sigma\text{-bimod}$. The following notion was introduced by B. Vallette [37–39].

DEFINITION 11.1 **Properads** are algebras over the triple $\Gamma_c : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.

A properad is therefore a Σ -bimodule with operations that determine coherent contractions along connected graphs. A biased definition of properads is given

in [37–39]. Since Γ_c is a subtriple of Γ_P , each PROP is automatically also a properad. Therefore one may speak about the *endomorphism properad* $\mathcal{E}nd_V$ and define *algebras* over a properad P as properad homomorphisms $\rho : P \rightarrow \mathcal{E}nd_V$. Algebras over other versions of PROPs recalled below can be defined in a similar way.

EXAMPLE 11.1 *Associative bialgebras reviewed in Example 10.3 are algebras over the properad B defined (tautologically) as the quotient of the free properad $\Gamma_c(\curlywedge, \curlyvee)$ by the properadic ideal generated by the elements listed in (10.3). We leave as an exercise to describe the sub-basis of (10.5) that span $B(m, n)$, $m, n \geq 1$.*

The following slightly artificial structure exists over PROPs but not over properads. It consists of a ‘multiplication’ $\mu = \curlywedge : V \otimes V \rightarrow V$, a ‘comultiplication’ $\Delta = \curlyvee : V \rightarrow V \otimes V$ and a linear map $f = \spadesuit : V \rightarrow V$ satisfying $\Delta \circ \mu = f \otimes f$ or, diagrammatically

$$\curlywedge = \spadesuit \spadesuit.$$

This structure cannot be a properad algebra because the graph on the right hand side of the above display is not connected.

Properads are still huge objects. The first really small version of PROPs were dioperads introduced in 2003 by W.L. Gan [45]. As a motivation for his definition, consider the following:

EXAMPLE 11.2 *A **Lie bialgebra** is a vector space V with a Lie algebra structure $[-, -] = \curlywedge : V \otimes V \rightarrow V$ and a Lie diagonal $\delta = \curlyvee : V \rightarrow V \otimes V$. We assume that $[-, -]$ and δ are related by*

$$\delta[a, b] = \sum ([a_{(1)}, b] \otimes a_{(2)} + [a, b_{(1)}] \otimes b_{(2)} + a_{(1)} \otimes [a_{(2)}, b] + b_{(1)} \otimes [a, b_{(2)}]) \quad (11.2)$$

for any $a, b \in V$, with the Sweedler notation $\delta a = \sum a_{(1)} \otimes a_{(2)}$ and $\delta b = \sum b_{(1)} \otimes b_{(2)}$.

Lie bialgebras are governed by the PROP $\text{LieB} = \Gamma(\curlywedge, \curlyvee)/\mathfrak{l}_{\text{LieB}}$, where $\curlywedge$ and $\curlyvee$ are now antisymmetric and $\mathfrak{l}_{\text{LieB}}$ denotes the ideal generated by

$$\begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \diagup \quad \diagdown \\ \text{1} \quad \text{2} \quad \text{3} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \diagdown \quad \diagup \\ \text{1} \quad \text{2} \quad \text{3} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \diagup \quad \diagup \\ \text{1} \quad \text{2} \quad \text{3} \end{array}, \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \diagdown \quad \diagdown \\ \text{1} \quad \text{2} \quad \text{3} \end{array} + \begin{array}{c} \text{2} \quad \text{3} \quad \text{1} \\ \diagdown \quad \diagdown \\ \text{1} \quad \text{2} \quad \text{3} \end{array} + \begin{array}{c} \text{3} \quad \text{1} \quad \text{2} \\ \diagdown \quad \diagdown \\ \text{1} \quad \text{2} \quad \text{3} \end{array} \quad \text{and} \quad \begin{array}{c} \text{1} \quad \text{2} \\ \diagdown \quad \diagup \\ \text{1} \quad \text{2} \end{array} - \begin{array}{c} \text{1} \quad \text{2} \\ \diagup \quad \diagdown \\ \text{1} \quad \text{2} \end{array} - \begin{array}{c} \text{1} \quad \text{2} \\ \diagdown \quad \diagdown \\ \text{1} \quad \text{2} \end{array} + \begin{array}{c} \text{2} \quad \text{1} \\ \diagup \quad \diagdown \\ \text{1} \quad \text{2} \end{array} + \begin{array}{c} \text{2} \quad \text{1} \\ \diagdown \quad \diagup \\ \text{1} \quad \text{2} \end{array}, \quad (11.3)$$

with labels indicating the corresponding permutations of the inputs and outputs.

We observe that all graphs in (11.3) are not only connected as demanded for properads, but also simply-connected. This suggests considering the full subcategory $\mathbf{UGr}_D(m, n)$ of $\mathbf{UGr}(m, n)$ consisting of *connected simply-connected* graphs and the related triple

$$\Gamma_D(E)(m, n) := \operatorname{colim}_{G \in \mathbf{UGr}_D(m, n)} E(G), \quad m, n \geq 0. \quad (11.4)$$

DEFINITION 11.2 *Dioperads are algebras over the triple $\Gamma_D : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.*

A biased definition of dioperads can be found in [45]. As observed by T. Leinster, dioperads are more or less equivalent to polycategories, in the sense of [46], with one object. Lie bialgebras reviewed in Example 11.2 are algebras over a dioperad. Another important class of dioperad algebras is recalled in:

EXAMPLE 11.3 *An **infinitesimal bialgebra** [47] (called in [48] a **mock bialgebra**) is a vector space V with an associative multiplication $\cdot : V \otimes V \rightarrow V$ and a coassociative comultiplication $\Delta : V \rightarrow V \otimes V$ such that*

$$\Delta(a \cdot b) = \sum (a_{(1)} \otimes a_{(2)} \cdot b + a \cdot b_{(1)} \otimes b_{(2)})$$

for any $a, b \in V$. It is easy to see that the axioms of infinitesimal bialgebras are encoded by the following simply connected graphs:

$$\begin{array}{c} \diagup \! \! \diagdown - \diagdown \! \! \diagup, \quad \nabla - \nabla \quad \text{and} \quad \diagdown - \begin{array}{c} \diagup \! \! \diagdown \\ \diagdown \! \! \diagup \end{array} - \begin{array}{c} \diagdown \! \! \diagup \\ \diagup \! \! \diagdown \end{array}. \end{array}$$

Observe that associative bialgebras recalled in Example 10.1 cannot be defined over dioperads, because the rightmost graph in (10.3) is not simply connected. The following proposition, which should be compared to Proposition 6.1, shows that dioperads are of the same size as operads.

PROPOSITION 11.1 *Let $E = \{E(m, n)\}_{m, n \geq 0}$ be a Σ -bimodule such that*

$$E(m, n) = 0 \text{ for } m + n \leq 2 \quad (11.5)$$

and that $E(m, n)$ is finite-dimensional for all remaining m, n . Then the components $\Gamma_D(E)(m, n)$ of the free dioperad $\Gamma_D(E)$ are finite-dimensional, for all $m, n \geq 0$.

The proof, similar to the proof of Proposition 6.1, is based on the observation that the assumption (11.5) reduces the colimit (11.4) to a summation over reduced trees (trees whose all vertices have at least three adjacent edges).

An important problem arising in connection with deformation quantization is to find a reasonably small, explicit cofibrant resolution of the bialgebra PROP $\mathbf{B}$. Here by a resolution we mean a differential graded PROP $\mathbf{R}$ together with a homomorphism $\beta : \mathbf{R} \rightarrow \mathbf{B}$ inducing a homology isomorphism. Cofibrant in this context means that $\mathbf{R}$ is of the form $(\Gamma_{\mathbf{P}}(E), \partial)$, where the generating Σ -bimodule E decomposes as $E = \bigoplus_{n \geq 0} E_n$ and the differential decreases the filtration, that is

$$\partial(E_n) \subset \Gamma_{\mathbf{P}}(E)_{<n}, \text{ for each } n \geq 0,$$

where $\Gamma_{\mathbf{P}}(E)_{<n}$ denotes the sub-PROP of $\Gamma(E)$ generated by $\bigoplus_{j < n} E_j$. This notion is an PROPic analog of the Koszul-Sullivan algebra in rational homotopy theory [49]. Several papers devoted to finding $\mathbf{R}$ appeared recently [42, 50–55]. The approach of [56] is based on the observation that $\mathbf{B}$ is a deformation, in the sense explained below, of the PROP describing structures recalled in the following:

DEFINITION 11.3 A *half – bialgebra* or simply a $\frac{1}{2}$ *bialgebra* is a vector space V with an associative multiplication $\mu : V \otimes V \rightarrow V$ and a coassociative comultiplication $\Delta : V \rightarrow V \otimes V$ that satisfy

$$\Delta(u \cdot v) = 0, \text{ for each } u, v \in V. \quad (11.6)$$

We chose this strange name because (11.6) is indeed one half of the compatibility relation (10.1) of associative bialgebras. $\frac{1}{2}$ bialgebras are algebras over the PROP

$$\frac{1}{2}\mathbf{B} := \Gamma(\wedge) / (\wedge = \wedge, \vee = \vee, \bowtie = 0).$$

Now define, for a formal variable t , $\mathbf{B}_t$ to be the quotient of the free PROP $\Gamma(\wedge, \vee)$ by the ideal generated by

$$\wedge = \wedge, \vee = \vee, \bowtie = t \cdot \text{diagram}.$$

Thus $\mathbf{B}_t$ is a one-parametric family of PROPs with the property that $\mathbf{B}_0 = \frac{1}{2}\mathbf{B}$. At a generic t , $\mathbf{B}_t$ is isomorphic to the bialgebra PROP $\mathbf{B}$. In other words, the PROP for bialgebras is a deformation of the PROP for $\frac{1}{2}$ bialgebras. According to general

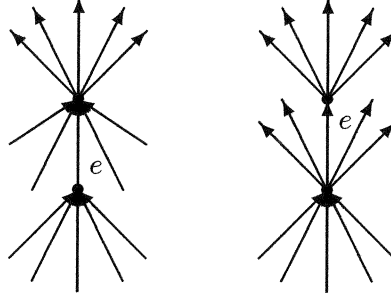


Figure 21: Edges allowed in a $\frac{1}{2}$ graph.

principles of homological perturbation theory [57], one may try to construct the resolution $\mathbf{R}$ as a perturbation of a cofibrant resolution $\frac{1}{2}\mathbf{R}$ of the PROP $\frac{1}{2}\mathbf{B}$. Since $\frac{1}{2}\mathbf{B}$ is simpler than $\mathbf{B}$, one may expect that resolving $\frac{1}{2}\mathbf{B}$ would be a simpler task than resolving $\mathbf{B}$.

For instance, one may realize that $\frac{1}{2}$ bialgebras are algebras over a dioperad $\frac{1}{2}\mathbf{B}$, use [45] to construct a resolution $\frac{1}{2}\mathbf{R}$ of the dioperad $\frac{1}{2}\mathbf{B}$, and then take $\frac{1}{2}\mathbf{R}$ to be the PROP generated by $\frac{1}{2}\mathbf{R}$. More precisely, one denotes

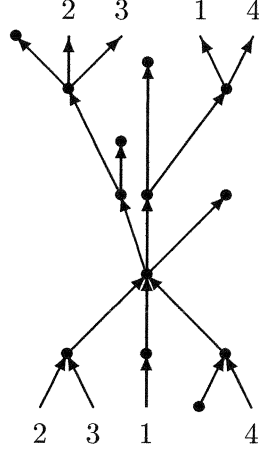
$$F_1 : \mathbf{diOp} \rightarrow \mathbf{PROP} \quad (11.7)$$

the left adjoint to the forgetful functor $\mathbf{PROP} \xrightarrow{\square_1} \mathbf{diOp}$ and defines $\frac{1}{2}\mathbf{R} := F_1(\frac{1}{2}\mathbf{R})$.

The problem is that we do not know whether the functor F_1 is exact, so it is not clear if $\frac{1}{2}\mathbf{R}$ constructed in this way is really a resolution of $\frac{1}{2}\mathbf{B}$. To get around this subtlety, M. Kontsevich observed that $\frac{1}{2}$ bialgebras live over a version of PROPs which is smaller than dioperads. It can be defined as follows.

Let an (m, n) - $\frac{1}{2}$ graph be a connected simply-connected directed (m, n) -graph whose each edge e has the following property: either e is the unique outgoing edge of its initial vertex or e is the unique incoming edge of its terminal vertex, see Figure 21. An example of an (m, n) - $\frac{1}{2}$ graph is given in Figure 22. Let $\mathbf{Gr}_{\frac{1}{2}}(m, n)$ be the category of (m, n) - $\frac{1}{2}$ graphs and their isomorphisms. Define a triple $\Gamma_{\frac{1}{2}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ by

$$\Gamma_{\frac{1}{2}}(E)(m, n) := \operatorname{colim}_{G \in \mathbf{Gr}_{\frac{1}{2}}(m, n)} E(G), \quad m, n \geq 0. \quad (11.8)$$

Figure 22: A graph from $\text{Gr}_{\frac{1}{2}}(4, 4)$.

DEFINITION 11.4 A $\frac{1}{2}\text{PROP}$ (called a **meager PROP**) is an algebra over the triple $\Gamma_{\frac{1}{2}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.

A biased definition of $\frac{1}{2}\text{PROPs}$ can be found in [40, 56]. We followed the convention that $\frac{1}{2}\text{PROPs}$ do not have units; the unital version of $\frac{1}{2}\text{PROPs}$ can be defined in an obvious way, compare also the remarks in [56].

EXAMPLE 11.4 $\frac{1}{2}\text{bialgebras}$ are algebras over a $\frac{1}{2}\text{PROP}$ which we denote $\frac{1}{2}\mathbf{b}$. Another example of structures that can be defined over $\frac{1}{2}\text{PROPs}$ are Lie $\frac{1}{2}\text{bialgebras}$ consisting of a Lie algebra bracket $[-, -] : V \otimes V \rightarrow V$ and a Lie diagonal $\delta : V \rightarrow V \otimes V$ satisfying one-half of (11.2):

$$\delta[a, b] = 0.$$

Let us denote by

$$F : \frac{1}{2}\text{PROP} \rightarrow \text{PROP}$$

the left adjoint to the forgetful functor $\text{PROP} \xrightarrow{\square} \frac{1}{2}\text{PROP}$ from the category of PROPs to the category of $\frac{1}{2}\text{PROPs}$. M. Kontsevich observed that, in contrast to $F_1 : \mathbf{diOp} \rightarrow \text{PROP}$ in (11.7), F is a *polynomial* functor, which immediately implies the following important theorem [40].

THEOREM 11.1 *The functor $F : \frac{1}{2}\mathbf{PROP} \rightarrow \mathbf{PROP}$ is exact.*

Now one may take a resolution $\frac{1}{2}\mathbf{r}$ of the $\frac{1}{2}\mathbf{PROP} \frac{1}{2}\mathbf{b}$ and put $\frac{1}{2}\mathbf{R} := F(\frac{1}{2}\mathbf{r})$. Theorem 11.1 guarantees that $\frac{1}{2}\mathbf{R}$ defined in this way is indeed a resolution of the $\mathbf{PROP} \frac{1}{2}\mathbf{B}$. Let us mention that there are also other structures invented to study resolutions of the $\mathbf{PROP} \mathbf{B}$, as $\frac{2}{3}\mathbf{PROPs}$ of Shoikhet [53], matrons of Saneblidze and Umble [50], or special $\mathbf{PROPs}$ considered in [56].

The constructions reviewed in this Section can be organized into the following chain of inclusions of full subcategories:

$$\mathbf{Oper} \subset \frac{1}{2}\mathbf{PROP} \subset \mathbf{diOp} \subset \mathbf{Proper} \subset \mathbf{PROP}.$$

The general scheme behind all these constructions is the following. We start by choosing a subgroupoid $\mathbf{SGr} = \bigsqcup_{m,n \geq 0} \mathbf{SGr}(m, n)$ of $\mathbf{Gr} := \bigsqcup_{m,n \geq 0} \mathbf{Gr}(m, n)$ (or a subgroupoid of $\mathbf{UGr} := \bigsqcup_{m,n \geq 0} \mathbf{UGr}(\mathbf{m}, \mathbf{n})$ if we want units). Then we define a functor $\Gamma_{\mathbf{S}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ by

$$\Gamma_{\mathbf{S}}(E)(m, n) := \operatorname{colim}_{G \in \mathbf{SGr}(m, n)} E(G), \quad m, n \geq 0.$$

It is easy to see that $\Gamma_{\mathbf{S}}$ is a subtriple of the $\mathbf{PROP}$ triple $\Gamma_{\mathbf{P}}$ if and only if the following two conditions are satisfied:

- (i) the groupoid $\mathbf{SGr}$ is *hereditary* in the sense that, given a graph from $\mathbf{SGr}$ with vertices decorated by graphs from $\mathbf{SGr}$, then the graph obtained by ‘forgetting the braces’ again belongs to $\mathbf{SGr}$, and
- (ii) $\mathbf{SGr}$ contains all directed corollas.

Hereditariness (i) is necessary for $\Gamma_{\mathbf{S}}$ to be closed under the triple multiplication of $\Gamma_{\mathbf{P}}$ while (ii) guarantees that $\Gamma_{\mathbf{S}}$ has an unit. Plainly, all the three choices used above – $\mathbf{UGr}_c$, $\mathbf{UGr}_D$ and $\mathbf{Gr}_{\frac{1}{2}}$ – satisfy the above assumptions. Let us mention that one may modify the definition of $\mathbf{PROPs}$ also by *enlarging* the category $\mathbf{Gr}(m, n)$, as was done for *wheeled PROPs* in [58]. Pasting schemes and the corresponding structures reviewed in this article are listed in Figure 23.

Pasting schemes	corresponding structures
rooted trees	non-unital operads
May's trees	non-unital May's operads
extended rooted trees	operads
cyclic trees	non-unital cyclic operads
extended cyclic trees	cyclic operads
stable labeled graphs	modular operads
extended directed graphs	PROPs
extended connected directed graphs	properads
extended connected 1-connected dir. graphs	dioperads
$\frac{1}{2}$ graphs	$\frac{1}{2}$ PROPs

Figure 23: Pasting schemes and the structures they define.

12 Sums over Trees

In this Section we prove basic formal identities for certain infinite sums (partition functions) taken over graphs of various topological types. The simplest “Euler product” identity relates sums over not necessarily connected graphs to those over connected ones. Summation over trees is interpreted as a calculation of the critical value of a formal potential. Finally, summation over graphs of arbitrary topology is interpreted as the perturbation series for a formal Feynman integral.

12.1 Application to sums over graphs

DEFINITION 12.1 *Let $(\mathcal{E}, \circ)$ be a symmetric monoidal category with the identity object $\mathbf{1}$ satisfying the following conditions.*

a) $\mathcal{E}$ has a countable set of isomorphism classes of objects. Every object has a finite automorphism group.

b) Every object of $\mathcal{E}$ is isomorphic to a product $\bigcirc_i \pi_i^{a_i}$ where π_i are indecomposable with respect to $\circ$ (“primes”), $\pi_i^{a_i}$ is the $\circ$ -product of a_i copies of π_i , and

$\pi_i \neq \pi_j$ for $i \neq j$. This product is defined uniquely up to permutation of factors.

c) We have

$$|\text{Aut } \bigcirc_i \pi_i^{a_i}| = \prod_i a_i! |\text{Aut } \pi_i|^{a_i}, \quad (12.1)$$

in particular, $|\text{Aut } (\mathbf{1})| = 1$.

In addition let R be a commutative topological ring and let $w : \text{Ob } \mathcal{E} \rightarrow R$ be a weight function depending only on the isomorphism class of the object and multiplicative: $w(\sigma \circ \tau) = w(\sigma)w(\tau)$.

THEOREM 12.1 *If the sums and products involved absolutely converge, we have*

$$\prod_{\{\pi\}/(iso)} \exp \frac{w(\pi)}{|\text{Aut } \pi|} = \sum_{\sigma \in \text{Ob } \mathcal{E}/(iso)} \frac{w(\sigma)}{|\text{Aut } \sigma|} = \exp \left(\sum_{\{\pi\}/(iso)} \frac{w(\pi)}{|\text{Aut } \pi|} \right). \quad (12.2)$$

Proof. We have

$$\prod_{\{\pi\}/(iso)} \exp \frac{w(\pi)}{|\text{Aut } \pi|} = \prod_{\{\pi\}/(iso)} \sum_{a=0}^{\infty} \frac{w(\pi)^a}{a! |\text{Aut } \pi|^a},$$

and it remains to apply (12.1). ■

Throughout this Section, we will take for $\mathcal{E}$ various categories of finite graphs, $\circ$ will denote the disjoint sum, and “primes” π will be connected graphs. Property (12.1) will be evident from the definition of isomorphisms. The second equality in (12.2) says that a weighted sum taken over all graphs can be obtained by exponentiation from the similar sum taken only over connected graphs.

We will now introduce a family of weights which will be called standard.

DEFINITION 12.2 *A **standard weight** on a category of finite graphs is defined by the following choices:*

- a) A set of “colors” A , finite or countable.
- b) A family of symmetric tensors $C_{a_1, \dots, a_k}$, $k = 1, 2, \dots$, whose subscripts belong to A and coordinates belong to a topological commutative ring R .
- c) A symmetric tensor g^{ab} with the same properties. The matrix (g^{ab}) must be invertible, and we put $(g_{ab}) = (g^{ab})^{-1}$.

In other words, we have a free R -module H with metric and a sequence of symmetric polynomials of all degrees on H expressed in terms of a basis indexed by A . (We can generalize this setting considering supercommutative R and $\mathbf{Z}_2$ -graded H .)

With these choices made, we put for a graph τ :

$$w(\tau) = \sum_{u: F_\tau \rightarrow A} \prod_{e \in E_\tau} g^{u(\partial e)} \prod_{v \in V_\tau} C_{u(F_\tau(v))}. \quad (12.3)$$

REMARK 12.1 *a) The expression ∂e in (12.3) means the set of two flags constituting the edge e . When a marking $u : F_\tau \rightarrow A$ is given, $u(\partial e) = \{a, b\}$ consists of two elements of A which produce g^{ab} . We can similarly define $C_{u(F_\tau(v))}$ thanks to the symmetry.*

b) If A is finite, the whole sum (12.3) is finite. Otherwise we have to postulate convergence already at this step. In our applications R will be a formal series ring. The multiplicativity of w with respect to disjoint union is evident.

c) Consider now the sum of type (12.2) with a standard weight:

$$Z_{\mathcal{E}}(w) = \sum_{\tau/(iso)} \frac{1}{|\text{Aut } \tau|} \sum_{u: F_\tau \rightarrow A} \prod_{e \in E_\tau} g^{u(\partial e)} \prod_{v \in V_\tau} C_{u(F_\tau(v))}. \quad (12.4)$$

*Such sums occur in some models of statistical and quantum physics. Coloring of flags corresponds to the picture of A types of particles propagating along the edges with amplitudes g^{ab} and interacting at vertices with amplitudes $C_{a_1, \dots, a_k}$. In this context, graphs are Feynman diagrams, and (12.4) can be called the **partition function**. The same formalism emerges in the general operadic context and in the topology of moduli spaces.*

12.2 Summation over trees

In this Subsection, we will calculate the partition function (12.3) in which the summation is taken over the set T of isomorphism classes of all (connected) trees *without tails* and having at least one edge.

We will treat here $C_{a_1, \dots, a_k}$ independent formal variables over a subring $R_0 \subset R$ containing g_{ab}, g^{ab} , and $\mathbf{Q}$. Then all our sums make sense as formal series.

We will express Z via a simpler formal function of auxiliary variables $\phi = \{\phi^a | a \in A\}$ independent over R :

$$\Phi(\phi) = -\frac{1}{2} \sum_{a,b} g_{ab} \phi^a \phi^b + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} C_{a_1, \dots, a_k} \phi^{a_1} \dots \phi^{a_k} . \quad (12.5)$$

Put $C^a = \sum_{b \in A} g^{ab} C_b$ and denote by $N \subset R$ the ideal generated by $C_{a_1, \dots, a_k}$ for all $k \geq 2$.

THEOREM 12.2 a) *The equations*

$$\frac{\partial \Phi(\phi)}{\partial \phi^a} = 0, \quad \forall a \in A, \quad (12.6)$$

admit the unique solution $\phi_0 = \{\phi_0^a\} \in R^A$ satisfying the condition

$$\phi_0^a \equiv C^a \pmod{N} . \quad (12.7)$$

b) *The partition function $Z = Z_T$ satisfies the differential equations*

$$\frac{\partial Z}{\partial C_a} = \phi_0^a, \quad a \in A, \quad (12.8)$$

and is the critical value of $\Phi(\phi)$:

$$Z = \Phi(\phi_0) . \quad (12.9)$$

REMARK 12.2 *The assumption that $C_{a_1, \dots, a_k}$ are independent formal variables is used several times in the statements and proofs: to locate the critical point ϕ_0 , to make sense of the left-hand side of (12.8), etc. However, when the identities (12.5) and (12.6) are proved in the formal context, they can be specialized to other topological rings R .*

Proof. a) Rewrite (12.6) as

$$\sum_{b \in A} g_{ab} \phi^b = C_a + \sum_{k \geq 2} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} \frac{\partial}{\partial \phi^a} (C_{a_1, \dots, a_k} \phi^{a_1} \dots \phi^{a_k}), \quad \forall a \in A, \quad (12.10)$$

that is,

$$\phi^a = C^a + \sum_{k \geq 2} \frac{1}{k!} \sum_{a_1, \dots, a_k, b \in A} g^{ab} \frac{\partial}{\partial \phi^a} (C_{a_1, \dots, a_k} \phi^{a_1} \dots \phi^{a_k}), \quad \forall a \in A. \quad (12.11)$$

Comparing (12.7) and (12.11) one sees that the critical point in question can be calculated by iterating (12.11). More precisely, consider the formal operator T mapping $\psi = (\psi^a | a \in A)$ to $(T^a(\psi) | a \in A)$ where

$$T^a(\psi) = \sum_{k \geq 2} \frac{1}{k!} \sum_{i=1}^k \sum_{a_1, \dots, a_k, b \in A} g^{ab} C_{a_1, \dots, a_k} \psi^{a_1} \dots \widehat{\psi^{a_i}} \dots \psi^{a_k} \delta_{a_i, b}. \quad (12.12)$$

The equation (12.11) can be rewritten as $\phi_0 = C + T(\phi_0)$ and solved by means of a version of the geometric progression formula

$$\phi_0 = C + T(C + T(C + T(C + \dots))). \quad (12.13)$$

The solution is clearly unique.

b) In order to make more transparent the formal structure of (12.13) as a sum over trees, we will consider the case when $A = \{*\}$ is a one-element set.

Put $g^{**} = g, g_{**} = g^{-1}, C_{*...*}$ (k subscripts) $= C_k, \phi^* = \phi$, and $C^1 = gC_1$. Then (12.12) becomes

$$T(\psi) = \sum_{k=1}^{\infty} \frac{gC_{k+1}}{k!} \psi^k$$

and (12.13) takes the form

$$\phi_0 = \sum_{k=0}^{\infty} \frac{gC_{k+1}}{k!} \left(\sum_{l=0}^{\infty} \frac{gC_{l+1}}{l!} \left(\sum_{m=0}^{\infty} \frac{gC_{m+1}}{m!} (gC_1 + \dots)^m \right)^l \right)^k. \quad (12.14)$$

Opening the brackets we will represent ϕ_0 sum of monomials in $\frac{gC_{i+1}}{i!}$. We will say that such a monomial has height $\leq N$ if it is a product of terms situated before the N -th opening bracket in (12.14) or directly after it (the terms of the latter type are gC_1).

E.g. the only monomial of height 0 is gC_1 . Monomials of height 1 are

$$\frac{gC_{k+1}}{k!} (gC_1)^k, \quad k \geq 1.$$

Monomials of height 2 are indexed by the families of integers $\{k_i; l_1, \dots, l_k\}$, $k \geq 1, l_i \geq 0$, each such family contributing

$$\frac{C_{k+1}}{k!} \frac{C_{l_1+1}}{l_1!} \dots \frac{C_{l_k+1}}{l_k!} (gC_1)^{l_1+\dots+l_k}. \quad (12.15)$$

To establish the general pattern, we need a definition. Consider a tree without tails τ . *The pinning* of τ is given by the choice of the following data:

a) The choice of a vertex $v_0 \in V_\tau$ with $|F_\tau(v_0)| = 1$ called *the root*.

Such choice determines a unique orientation of all flags (or edges) of τ such that the unique flag of v_0 is incoming and every vertex $v \neq v_0$ has exactly one outgoing flag.

Such choice determines a unique orientation of all flags (or edges) of τ such that the unique flag of v_0 is incoming and every vertex $v \neq v_0$ has exactly one outgoing flag.

b) A total ordering of all sets $V_\tau(k) \subset V_\tau$ where $V_\tau(k)$ denotes the set of all vertices of τ separated by k edges from v_0 . This total ordering must satisfy the following condition. Let $f_k : V_\tau(k+1) \rightarrow V_\tau(k)$ be the map “going along the outgoing edge to the next vertex”. Then f_k must be monotone with respect to the chosen orderings.

A *pinned tree* is a tree with pinning. An isomorphism of pinned trees is an isomorphism of trees compatible with orientation and pinning. *The height* of a pinned tree is $\max \{k | V_\tau(k+1) \neq \emptyset\}$.

A contemplation shows that there is a natural bijection between the isomorphism classes of pinned trees (τ, p) with $|E_\tau| \geq 1$ of height $\leq N$ and monomials of height $\leq N$ which can be directly obtained from (12.14). Moreover, various pinnings of the same τ generate the differently ordered but equal monomials which can be written in the form dependent only on τ :

$$\frac{1}{C_1} g^{|E_\tau|} \prod_{v \in V_\tau} C_{|v|} / (|v| - 1)!. \quad (12.16)$$

Now, the number of different pinnings of τ is $|T_\tau| \prod_{v \in V_\tau} (|v| - 1)!$ where T_τ is the set of potential roots and factorials count orderings of incoming edges. The automorphism group of τ effectively acts on the set of pinnings. Hence (12.16)

appears with the coefficient

$$|T_\tau| \prod_{v \in V_\tau} (|v| - 1)! / |\text{Aut } \tau| .$$

We now turn to the proof of (12.8) which for the one-element A becomes

$$\phi_0 = \frac{\partial}{\partial C_1} \left(\sum_{\tau} \frac{1}{|\text{Aut } \tau|} g^{|E_\tau|} \prod_{v \in V_\tau} C_{|v|} \right) . \quad (12.17)$$

In fact, the discussion above shows that the tree τ with all its pinnings contributes to ϕ_0 the term

$$\frac{|T_\tau| \prod_{v \in V_\tau} (|v| - 1)!}{C_1 |\text{Aut } \tau|} g^{|E_\tau|} \prod_{v \in V_\tau} \frac{C_{|v|}}{(|v| - 1)!} .$$

In view of (12.4), this is the same as the contribution of τ to $\frac{\partial Z}{\partial C_1}$. This gives (12.8).

We leave to the reader the discussion of the case $|A| \geq 1$.

To derive (12.9) from (12.8), consider both sides of (12.9) as formal series in $C_a, a \geq 1$. Their constant terms (value at $(C_a) = 0$) vanish. For Z , this follows from the fact that any tree in (12.4) has at least two vertices with $|v| = 1$. For ϕ_0 , this follows from (12.17). Hence it is sufficient to check that $\frac{\partial}{\partial C_a} Z = \frac{\partial}{\partial C_a} \Phi(\phi_0)$ for all $a \in A$. But we have

$$\frac{\partial}{\partial C_a} \Phi(\phi_0) = \sum_{b \in A} \frac{\partial \Phi}{\partial \phi^b}(\phi_0) \frac{\partial \phi_0^b}{\partial C_a} + \frac{\partial \Phi}{\partial C_a}(\phi_0) .$$

The first sum vanishes because $(dS)(\phi_0) = 0$, and the second term equals ϕ_0^a because of (12.5). It remains to apply (12.8). ■

12.3 Summation over graphs of arbitrary topology

We will now study the partition function (12.4) for more general graphs, keeping the same assumptions about the coefficient ring R and tensors C, g as in Subsections 12.1 and 12.2. In order to keep track of the Euler characteristic of the graphs, we extend R to the Laurent formal series ring $R_\lambda = R((\lambda^{-1}))$.

DEFINITION 12.3 *An R_λ -linear functional*

$$\langle \cdot \rangle : R_\lambda[[\phi]] \rightarrow R_\lambda$$

is called $\lambda^{-1}g$ -Gaussian (mean value) if it is (λ^{-1}, ϕ) -adically continuous, and

$$\langle \exp(\lambda^{-1} \sum_a C_a \phi^a) \rangle = \exp((2\lambda^{-1}) \sum_a C_a g^{ab} C_b) . \quad (12.18)$$

LEMMA 12.1 (**Wick**) *If (C_a) are independent variables over R_λ , then we have:*

$$a) \langle \phi^{a_1} \dots \phi^{a_n} \rangle = 0 \text{ for } n \equiv 1 \pmod{2}.$$

$$b) \langle \phi^a \phi^b \rangle = \lambda g^{ab}.$$

c) $\langle \phi^{a_1} \dots \phi^{a_{2m}} \rangle = \lambda^m \sum g^{a_{i_1} a_{j_1}} \dots g^{a_{i_m} a_{j_m}}$ where the summation is taken over all unordered partitions of $\{1, \dots, 2m\}$ into m unordered pairs $\{i_1, j_1\}, \dots, \{i_m, j_m\}$ (pairings).

Conversely, if a (λ^{-1}, ϕ) -adically continuous functional $\langle \cdot \rangle$ satisfies a), b), c), then it is $\lambda^{-1}g$ -Gaussian.

Proof. We have

$$\langle \exp(\lambda^{-1} \sum_a C_a \phi^a) \rangle = \sum_{n=0}^{\infty} \frac{1}{\lambda^n n!} \sum_{a_1, \dots, a_n \in A} C_{a_1} \dots C_{a_n} \langle \phi^{a_1} \dots \phi^{a_n} \rangle ,$$

$$\exp((2\lambda)^{-1} \sum_a C_a g^{ab} C_b) = \sum_{m=0}^{\infty} \frac{1}{2^m \lambda^m m!} \sum_{a_i, b_i \in A} C_{a_1} C_{b_1} \dots C_{a_m} C_{b_m} g^{a_1 b_1} \dots g^{a_m b_m} .$$

Comparing the coefficients, we get the lemma. ■

Now put

$$\Phi_0(\phi) = -\frac{1}{2} \sum_{a, b \in A} g_{ab} \phi^a \phi^b, \quad \Phi_1(\phi) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} C_{a_1, \dots, a_k} \phi^{a_1} \dots \phi^{a_k} ,$$

and denote by $w(\tau)$ the weight function (12.3).

Let Γ be the set of (isomorphism classes of) all finite graphs without tails, not necessarily connected, including the empty graph, and Γ_0 the subset of connected non-empty graphs. Let $\{\cdot\}$ be the $\lambda^{-1}g$ -Gaussian mean value. Denote by $\chi(\tau)$ the Euler characteristic of $\|\tau\|$.

THEOREM 12.3 *We have*

$$\sum_{\tau \in \Gamma} \frac{\lambda^{\chi(\tau)}}{|\text{Aut } \tau|} w(\tau) = \langle \exp(\lambda^{-1} \Psi_1(\phi)) \rangle, \quad (12.19)$$

$$\sum_{\tau \in \Gamma_0} \frac{\lambda^{\chi(\tau)}}{|\text{Aut } \tau|} w(\tau) = \log \langle \exp(\lambda^{-1} \Psi_1(\phi)) \rangle. \quad (12.20)$$

Proof. The second equality follows from the first one in view of (12.2) and the additivity of the Euler characteristic with relation to the disjoint union.

Let us now calculate the right-hand side of (12.19). By definition, it is

$$\left\langle \sum_{n=0}^{\infty} \lambda^{-n} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} C_{a_1, \dots, a_k} \phi^{a_1} \dots \phi^{a_k} \right)^n \right\rangle. \quad (12.21)$$

Choose some $(n; k_1, \dots, k_n)$. A typical monomial in the decomposition of (12.21) will be

$$\lambda^{-1} \frac{1}{n!} \prod_{i=1}^n \frac{1}{k_i!} C_{a_1^{(i)}, \dots, a_{k_i}^{(i)}} \langle \prod_{i=1}^n \phi^{a_1^{(i)}} \dots \phi^{a_{k_i}^{(i)}} \rangle. \quad (12.22)$$

It vanishes if $k_1 + \dots + k_n$ is odd. Otherwise, in view of Wick's Lemma (12.22) can be rewritten as

$$\lambda^{-n + \frac{1}{2} \sum k_i} \frac{1}{n!} \prod_{i=1}^n \frac{1}{k_i!} C_{a_1^{(i)}, \dots, a_{k_i}^{(i)}} \left(\sum g^{a_{l_1}^{(i_1)} a_{m_1}^{(j_1)}} \dots g^{a_{l_r}^{(i_r)} a_{m_r}^{(j_r)}} \right), \quad (12.23)$$

where $r = \frac{1}{2} \sum k_i$ and the inner sum is taken over all pairings of the set of ordered pairs $F = \bigcup_{i=1}^n \{(i, 1), \dots, (i, k_i)\}$.

Construct the family of graphs τ whose set of flags is $F_\tau := F$, $V_\tau = \{1, \dots, n\}$, $\partial_\tau(i, l) = i$, and involutions bijectively correspond to various pairings in (12.23). If we color the flags of one such graph by the map $F_\tau \rightarrow A : (i, l) \mapsto a_l^{(i)}$, then the sum over all pairings will produce the same monomials as in (12.4). It remains to do the accurate bookkeeping in order to identify the coefficients.

The graphs constructed above bijectively correspond to all elements of Γ . In fact, a choice of $(n; k_1, \dots, k_n)$ determines the number of vertices of any valence, and the choice of a pairing determines which pairs of flags become edges ($n = 0$

produces the empty graph). Moreover, a non-empty graph comes thus equipped with a total ordering of its vertices and all sets of flags belonging to one vertex. The sum over graphs does not take care of these orderings. The group $\text{Aut } \tau$ effectively acts on the whole set of them consisting of $n! \prod_{i=1}^n k_i!$ elements. Summing over isomorphism classes, we may replace the numerical coefficient in (12.23) by $|\text{Aut } \tau|^{-1}$.

Finally,

$$-n + \frac{1}{2} \sum_{i=1}^n k_i = -|V_\tau| + |E_\tau| = \chi(\tau) .$$

■

13 Generating Functions

In this Section we calculate several generating functions related to moduli spaces and quantum cohomology, first representing them as sums over trees of the type treated in Section 12.

13.1 Virtual Poincaré polynomial

Let Y be an algebraic variety over $\mathbf{C}$, possibly non-smooth and non-compact. Following [59] we denote by $P_Y(q)$ the virtual Poincaré polynomial of Y which is uniquely defined by the following properties.

a) If Y is smooth and compact, then

$$P_Y(q) = \sum_j \dim H^j(Y) q^j . \quad (13.1)$$

In particular

$$\chi(Y) = P_Y(-1) . \quad (13.2)$$

b) If $Y = \coprod_i Y_i$ is a finite union of pairwise disjoint locally closed strata, then

$$P_Y(q) = \sum_i P_{Y_i}(q) . \quad (13.3)$$

c) $P_{Y \times Z}(q) = P_Y(q) P_Z(q)$. It follows that if Y is a fibration over base B with fiber F locally trivial in the Zariski topology, then $P_Y(q) = P_B(q) P_F(q)$.

A definition of $P_Y(q)$ can be given using the weight filtration on the cohomology with compact support:

$$P_Y(q) = \sum_{i,j} (-1)^{i+j} \dim(\mathrm{gr}_W^j H_c^i(Y, \mathbf{Q})) q^j . \quad (13.4)$$

13.2 Generating function for moduli spaces of genus zero

We put

$$\varphi(q, t) := t + \sum_{n=2}^{\infty} P_{\overline{M}_{0,n+1}}(q) \frac{t^n}{n!} \in \mathbf{Q}[q][[t]] , \quad (13.5)$$

$$\chi(t) := \varphi(-1, t) = t + \sum_{n=2}^{\infty} \chi(\overline{M}_{0,n+1}) \frac{t^n}{n!} \in \mathbf{Q}[[t]] . \quad (13.6)$$

THEOREM 13.1 *a) $\varphi(q, t)$ is the unique root in $t + t^2 \mathbf{Q}[q][[t]]$ of any one of the following functional/differential equations in t with parameter q :*

$$(1 + \varphi)^{q^2} = q^4 \varphi - q^2(q^2 - 1)t + 1 , \quad (13.7)$$

$$(1 + q^2 t - q^2 \varphi) \varphi_t = 1 + \varphi . \quad (13.8)$$

b) χ is the unique root in $t + t^2 \mathbf{Q}[[t]]$ of any one of the similar equations

$$(1 + \chi) \log(1 + \chi) = 2\chi - t , \quad (13.9)$$

$$(1 + t - \chi) \chi_t = 1 + \chi . \quad (13.10)$$

Equation (13.8) is equivalent to the following recursive formulas for the Poincaré polynomials. Put $p_n = p_n(q) = P_{\overline{M}_{0,n+1}}/n!$.

COROLLARY 13.1 *We have for $n \geq 1$:*

$$(n+1)p_{n+1} = p_n + q^2 \sum_{\substack{i+j=n+1 \\ i \geq 2}} j p_i p_j , \quad (13.11)$$

$$P_{\overline{M}_{0,n+2}}(q) = P_{\overline{M}_{0,n+1}}(q) + q^2 \sum_{\substack{i+j=n+1 \\ i \geq 2}} \binom{n}{i} P_{\overline{M}_{0,i+1}}(q) P_{\overline{M}_{0,j+1}}(q) . \quad (13.12)$$

From (13.10) one sees that the function inverse to χ has a critical point at $t = e - 2$. From this one can derive the following asymptotical formula:

$$\chi(\overline{M}_{0,n+1}) \cong \frac{1}{\sqrt{n}} \left(\frac{n}{e^2 - 2e} \right)^{n-\frac{1}{2}}.$$

In order to prove Theorem 13.1, we will first apply the additivity formula (13.3) to the open boundary strata of $\overline{M}_{0,n}$ and then use Theorem 12.2. However, the classes of trees involved in the labeling of stable curves, on the one hand, and the summation formula (12.9), on the other, are slightly different: we need tails in the first problem and do not allow them in the second. In order to unify the combinatorial pictures, and *only in this Section*, we will eliminate tails by putting end-point vertices on them. This will lead to the following temporary modification of our conventions:

A tree without tails is called *stable* if $|v| \neq 2$ for all vertices v . If $|v| = 1$ we call v an end vertex. Let V_τ^1 be the set of end vertices. An n -marking of τ is a bijection $\mu : V_\tau^1 \rightarrow \{1, \dots, n\}$. We also put $V_\tau^0 = V \setminus V_\tau^1$ and refer to it as the set of interior vertices.

Now let $(C; x_1, \dots, x_n)$ be a stable compact connected curve of arithmetical genus zero with $n \geq 3$ labeled non-singular points. The combinatorial structure of this curve is described by the following stable tree with n -marking $(\tau, \mu) : V_\tau^0 = \{\text{irreducible components of } C\}$, $V_\tau^1 = \{x_1, \dots, x_n\}$; $\mu : x_i \mapsto i$; an edge connects two interior vertices if the respective components of C have non-empty intersection; an edge connects an interior vertex to an end vertex if the respective point belongs to the respective component.

Denote now by $M(\tau, \mu) \subset \overline{M}_{0,n}$ the set of points parametrizing stable curves of the type (τ, μ) . If τ has only one interior vertex, $M(\tau, \mu) := M_{0,n}$ is the big cell. The following statement summarizes the main properties of these sets; for a proof, see [60],

PROPOSITION 13.1 *a) $M(\tau, \mu)$ is a locally closed subset of $\overline{M}_{0,n}$ depending only on (the isomorphism class of) (τ, μ) .*

b) $\overline{M}_{0,n}$ is the union of pairwise disjoint strata $M(\tau, \mu)$ for all marked stable n -trees (τ, μ) .

c) For any (τ, μ) ,

$$M(\tau, \mu) \cong \prod_{v \in V_\tau^0} M_{0,|v|} . \quad (13.13)$$

Notice that there exists exactly one stable tree $\bullet \text{---} \bullet$ which does not correspond to any stable curve.

We can now calculate Poincaré polynomials.

PROPOSITION 13.2 *We have*

$$P_{M(\tau, \mu)}(q) = \prod_{v \in V_\tau^0} P_{M_{0,|v|}}(q) , \quad (13.14)$$

$$P_{M_{0,k}}(q) = \binom{q^2 - 2}{k - 3} (k - 3)! . \quad (13.15)$$

Proof. (13.14) follows from (13.13) and the multiplicativity of Poincaré polynomials.

To prove (13.15), one can use the following geometric facts. First, the morphism $\pi : \overline{M}_{0,n+1} \rightarrow \overline{M}_{0,n}$ forgetting the last marked point is (canonically isomorphic to) the universal curve. Second, the boundary of the source consists of structure sections and fibers at infinity of the target. Therefore, over the big cell $M_{0,n}$ this morphism is a Zariski locally trivial fibration with fiber $\mathbf{P}^1$, and $\overline{M}_{0,n+1} = \pi^{-1}(M_{0,n}) \setminus \{\text{union of structure sections}\}$.

From the additivity of Poincaré polynomials it follows that

$$P_{M_{0,n+1}}(q) = P_{M_{0,n}}(q)P_{\mathbf{P}^1}(q) - nP_{M_{0,n}}(q) = (q^2 + 1 - n)P_{M_{0,n}}(q) .$$

Since $P_{M_{0,3}}(q) = 1$, we get (13.15). ■

Summarizing, we have for $n \geq 3$:

$$P_{\overline{M}_{0,n}}(q)t^n = \sum_{\substack{(\tau, \mu)/(iso) \\ |V_\tau^1| = n}} \prod_{v \in V_\tau^0} \binom{q^2 - 2}{|v| - 3} (|v| - 3)! \prod_{v \in V_\tau^1} t , \quad (13.16)$$

where t is a new formal variable, and the sum is taken over n -marked stable trees.

We want to present (13.5) as a partition function. Comparing (13.16) to (12.3) and (12.5), we are more or less compelled to choose $A = \{*\}$ (one element set),

$g^{**} = 1$, $C_* = t$, $C_{**} = 0$ (this gives weight zero to non-stable trees), and finally, denoting by C_k the component with $k \geq 3$ subscripts, we get

$$C_k = \binom{q^2 - 2}{k - 3} (k - 3)! . \quad (13.17)$$

In particular, we can forget about $u : F_\tau \rightarrow \{*\}$.

If $|V_\tau^1| = n$, the set of all n -markings of τ consists of $n!$ elements and is effectively acted upon by the group $\text{Aut } \tau$. We see finally that $\psi(q, t) = Z$ where

$$\psi(q, t) := \frac{t^2}{2!} + \sum_{n \geq 3} \frac{t^n}{n!} P_{\overline{M}_{0,n}}(q) , \quad (13.18)$$

$$Z := \sum_{\tau/(iso)} \frac{1}{|\text{Aut } \tau|} \prod_{v \in V_\tau} C_{|v|} . \quad (13.19)$$

The summation in (13.19) is now taken over all trees, and the term $t^2/2$ in (13.18) comes from the two-vertex tree.

We will now use (12.8) in order to calculate

$$\frac{\partial Z}{\partial t} = \frac{\partial \psi(q, t)}{\partial t} =: \varphi(q, t) .$$

From (12.5) and (13.17) one sees that

$$\Phi(\varphi) = -\frac{\varphi^2}{2} + t\varphi + \sum_{k \geq 3} C_k \frac{\varphi^k}{k!} = -\frac{\varphi^2}{2} + t\varphi + \sum_{k \geq 3} \binom{q^2 - 2}{k - 3} \frac{\varphi^k}{k(k-1)(k-2)} .$$

This can easily be summed. We need only the derivative.

For generic q we have

$$\frac{\partial}{\partial \varphi} \Phi(\varphi) = \frac{(1 + \varphi)^{q^2} - 1 - q^4 \varphi}{q^2(q^2 - 1)} + t , \quad (13.20)$$

and for $q = -1$,

$$\frac{\partial}{\partial \varphi} \Phi(\varphi) = (1 + \varphi) \log(1 + \varphi) - 2\varphi + t . \quad (13.21)$$

(4.21)

We see now that (13.7), resp. (13.9), are equations for the critical point $d_\varphi \Phi = 0$. Differentiating them in t and eliminating $(1 + \varphi)^{q^2}$, resp. $\log(1 + \varphi)$, we get (13.8), resp. (13.10).

13.3 Generating function for configuration spaces

Let X be a smooth compact algebraic variety. The configuration space $X[n]$, $n \geq 2$, is defined in [59] as the closure of its big cell $X^n \setminus (\bigcup_{i < j} \Delta_{ij})$ (Δ_{ij} is the diagonal $x_i = x_j$) in $X^n \times \prod_S \tilde{X}^S$, where S runs over subsets $S \subset \{1, \dots, n\}$, $|S| \geq 2$; X^S denotes the respective partial product of X 's, and $\tilde{X}^S$ is the blow up of the small diagonal Δ_S in X^S .

Every S determines a divisor at infinity $D(S) \subset X[n]$. Namely, let $\pi_S : X[n] \rightarrow X^S$ be the canonical projection. Then $\pi_S^{-1}(\Delta_S) = \bigcup_{T \supset S} D(T)$.

The natural stratification of $X[n]$ described in [59] consists of (open subsets of) intersections $\overline{X(\mathcal{S})} = \bigcap_{i=1}^r D(S_i)$ corresponding to sets $\mathcal{S} = \{S_1, \dots, S_r\}$ of subsets in $\{1, \dots, n\}$ called *nests*.

We put

$$\psi_X(q, t) = 1 + \sum_{n \geq 1} P_{X[n]}(q) \frac{t^n}{n!} \in \mathbf{Q}[q][[t]] ,$$

$$\chi_X(t) = \psi_X(-1, t) = 1 + \sum_{n \geq 1} \chi(X[n]) \frac{t^n}{n!} \in \mathbf{Q}[[t]] .$$

Put also

$$\kappa_m = \frac{q^{2m} - 1}{q^2 - 1} = P_{\mathbf{P}^{m-1}}(q) .$$

THEOREM 13.2 *Denote by $y^0 = y^0(g, t)$ the unique root in $t + t^2 \mathbf{Q}[q^2][[t]]$ of any one of the following equations:*

$$\kappa_m(1 + y^0)^{q^{2m}} = q^{2m}(q^{2m} + \kappa_m - 1)y^0 - q^{2m}(q^{2m} - 1)t + \kappa_m , \quad (13.22)$$

$$[q^{2m}t + 1 - (q^{2m} - 1 + \kappa_m)y^0]y_t^0 = 1 + y^0 . \quad (13.23)$$

Then we have in $\mathbf{Q}[q][[t]]$:

$$\psi_X(q, t) = (1 + y^0)^{P_X(q)} . \quad (13.24)$$

THEOREM 13.3 *Denote by $\eta = \eta(t)$ the unique root in $t + t^2 \mathbf{Q}[[t]]$ of any one of the following equations:*

$$m(1 + \eta) \log(1 + \eta) = (m + 1)\eta - t , \quad (13.25)$$

$$(t + 1 - m\eta)\eta_t = 1 + \eta . \quad (13.26)$$

Then we have in $\mathbf{Q}[[t]]$:

$$\chi_X(t) = (1 + \eta)^{\chi(X)} . \quad (13.27)$$

We start with combinatorics of the strata.

DEFINITION 13.1 *a) $\mathcal{S} = \{S_1, \dots, S_r\}$ is a nest (or n -nest) if $|S_i| \geq 2$ for all i , and either $S_i \subset S_j$ or $S_j \subset S_i$ for all i, j such that $S_i \cap S_j \neq \emptyset$.*

In particular, $\mathcal{S} = \emptyset$ is a nest, and $\mathcal{S} = \{S\}$ is a nest, if $|S| \geq 2$.

*b) A nest $\mathcal{S}$ is called **whole** (resp. **broken**) if $\{1, \dots, n\} \in \mathcal{S}$ (resp. $\{1, \dots, n\} \notin \mathcal{S}$).*

Denote by $X(\mathcal{S}) \subset \overline{X(\mathcal{S})} = \bigcap_{S \in \mathcal{S}} D(S)$ the subset of points not belonging to smaller closed strata. The following facts are proved in [59].

PROPOSITION 13.3 *a) For any $n \geq 2$ and n -nest $\mathcal{S}$, $X(\mathcal{S})$ is a locally closed subset of $X[n]$.*

b) $X[n]$ is the union of pairwise disjoint strata $X(\mathcal{S})$ for all n -nests $\mathcal{S}$.

Now we will show how to pass from nests to marked trees. As above, we consider a bijection $\mu : V_\tau^1 \rightarrow [1, \dots, n]$ as a part of the appropriate marking for our problem. The remaining data is supplied by choosing *orientations of all edges*.

DEFINITION 13.2 *A tree τ marked in this way is called **admissible** iff:*

- a) Every vertex of τ except one has exactly one incoming edge.*
- b) The exceptional vertex has only outgoing edges, and their number is ≥ 2 . This vertex is called the **source**.*
- c) All interior vertices with possible exception of the source have valency ≥ 3 .*

PROPOSITION 13.4 *The following maps are (1,1):*

$$\{\text{broken } n\text{-nests}\} \rightarrow \{\text{whole } n\text{-nests}\} \rightarrow \{\text{admissible marked } n\text{-trees}\}/(\text{iso}),$$

$$\mathcal{S} \mapsto \mathcal{S} \cup \{\{1, \dots, n\}\} \mapsto \tau(\mathcal{S}) = \tau(\mathcal{S} \cup \{\{1, \dots, n\}\}) .$$

Here τ is defined by its sets of vertices and edges: if $\mathcal{S} = \{S_1, \dots, S_r\}$, then

$$V_\tau = \{\tilde{S}_1, \dots, \tilde{S}_{n+r}\} := \{S_1, \dots, S_r, \{1\}, \dots, \{n\}\} ,$$

and an edge oriented from $\tilde{S}_i$ to $\tilde{S}_j$ connects these two vertices iff $\tilde{S}_j \subset \tilde{S}_i$ and no $\tilde{S}_k$ lies strictly in between these two subsets.

This is proved by direct observation. The following facts are worth mentioning.

- a) $\{1, \dots, n\}$ is the source of $\tau(\mathcal{S})$ for any $\mathcal{S}$.
- b) $\{1\}, \dots, \{n\}$ are all end vertices.
- c) $i \in S_j$ iff one can pass from $S_j \in V_\tau$ to $\{i\} \in V_\tau$ in τ by always going in the positive direction.

The reader is advised to convince him- or herself that the source has valency ≥ 2 and all other interior vertices have valency ≥ 3 .

Denote the source by s and the set of the remaining interior vertices by V_τ^0 .

PROPOSITION 13.5 ([59]) *The virtual Poincaré polynomials of the strata $X(\mathcal{S})$ are given by the following formulas (we add a formal variable t):*

If $\mathcal{S}$ is a broken n -nest, $s \in V_{\tau(\mathcal{S})}$:

$$t^n P_{X(\mathcal{S})}(q) = \binom{P_X(q)}{|s|} |s|! \times \prod_{v \in V_{\tau(\mathcal{S})}^0} \kappa_m \left(\frac{q^{2m} - 2}{|v| - 3} \right) (|v| - 3)! \times \prod_{v \in V_{\tau(\mathcal{S})}^1} t. \quad (13.28)$$

If $\mathcal{S}$ is a whole n -nest:

$$t^n P_{X(\mathcal{S})}(q) = P_X(q) \kappa_m \left(\frac{q^{2m} - 2}{|s| - 2} \right) (|s| - 2)! \quad (13.29)$$

$$\times \prod_{v \in V_{\tau(\mathcal{S})}^0} \kappa_m \left(\frac{q^{2m} - 2}{|v| - 3} \right) (|v| - 3)! \times \prod_{v \in V_{\tau(\mathcal{S})}^1} t.$$

Comparing (13.28) and (13.29) one sees that one can express the joint contribution of two nests corresponding to an admissible marked tree τ as a product of local weights corresponding to all vertices of τ . The local weight of the source will be

$$\binom{P_X(q)}{|s|} |s|! + P_X(q) \kappa_m \left(\frac{q^{2m} - 2}{|s| - 2} \right) (|s| - 2)!$$

and the remaining local weights coincide and depend only on the valency.

In order to find the appropriate standard weights of marked trees (summands in (12.3)), we make the following choices.

Put $A = \{+, -\}$. Interpret a mark $+$ (resp. $-$) on a flag as incoming (resp. outgoing) orientation of this flag. Thus, $f : F_\tau \rightarrow A$ is a choice of orientation of all flags.

Put $g^{+-} = g^{-+} = 1$, $g^{++} = g^{--} = 0$. This makes the standard weight of (τ, f) vanish unless all edges are unambiguously oriented by f .

Put $C_+ = t$ (see (13.22) and (13.23)) and $C_- = 0$. The last choice makes the standard weight vanish unless all end edges are oriented outwards.

Put $C_{+-} = C_{-+} = 0$. This excludes vertices of the type $\rightarrow \bullet \rightarrow$. Put also $C_{a_1, \dots, a_k} = 0$ if $\{+, +\} \subset \{a_1, \dots, a_k\}$. This eliminates vertices with ≥ 2 incoming edges.

For tensors with $k \geq 2$ minuses among the indices we put

$$C_{- \dots -} = \binom{P_X(q)}{k} k! + \kappa_m P_X(q) \binom{q^{2m} - 2}{k - 2} (k - 2)! \quad (13.30)$$

(because only the source has all outgoing edges), and

$$C_{+ - \dots -} = \kappa_m \binom{q^{2m} - 2}{k - 2} (k - 2)! \quad (13.31)$$

(cf. (13.28) and (13.29)).

The standard weight of a marked tree defined by this data again is independent on the part $\mu : V_\tau^1 \rightarrow \{1, \dots, n\}$ of the initial marking, which accounts for the factor $\frac{n!}{|\text{Aut } \tau|}$ below.

Summarizing, we put

$$\psi_X(q, t) := \sum_{n \geq 2} \frac{t^n}{n!} P_{X|n|}(q), \quad (13.32)$$

$$Z := \sum_{\tau/(iso)} \frac{1}{|\text{Aut } \tau|} \sum_{f: F_\tau \rightarrow \{+, -\}} \prod_{\alpha \in E_\tau} g^{f(\partial \alpha)} \prod_{v \in V_\tau} C_{f(\sigma v)}, \quad (13.33)$$

and get from the previous discussion

$$Z = \psi_X(q, t), \quad \frac{\partial}{\partial t} Z := \phi_x(q, t). \quad (13.34)$$

The arguments in the potential will be denoted $\varphi_+ = x$, $\varphi_- = y$. We see that the potential is

$$\Phi(x, y) = -xy + tx + \kappa_m \sum_{k=2}^{\infty} \binom{q^{2m} - 2}{k - 2} \frac{xy^k}{k(k-1)} +$$

$$+ \sum_{k=2}^{\infty} \binom{P_X(q)}{k} y^k + \kappa_m P_X(q) \sum_{k=2}^{\infty} \binom{q^{2m}-2}{k-2} \frac{y^k}{k(k-1)} \quad (13.35)$$

(we have two arguments x, y but only one $t = t_+$ because $C_- = 0$).

We must solve the system

$$\frac{\partial \Phi}{\partial x}|_{x^0, y^0} = \frac{\partial \Phi}{\partial y}|_{x^0, y^0} = 0, \quad (13.36)$$

and (12.8) then tells us that

$$\frac{\partial}{\partial t} Z = \varphi_X(q, t) = x^0. \quad (13.37)$$

Again, $\Phi(x, y)$ can be easily summed. To write down the functional equation, we need only the x -derivative which for general q is

$$\frac{\partial \Phi}{\partial x} = -y + t + \kappa_m \frac{(1+y)^{q^{2m}} - 1 - q^{2m}y}{q^{2m}(q^{2m}-1)}. \quad (13.38)$$

For $q = -1$:

$$\frac{\partial \Phi}{\partial x} = -y + t + m[(1+y)\log(1+y) - y]. \quad (13.39)$$

We now see that (13.22), resp. (13.25), are the equations defining y^0 . Taking the derivative in t we get (13.23) and (13.26). And since $\Phi(x, y)$ is linear in x , the vanishing of the y -derivative provides an explicit expression of x^0 via y^0 :

$$\varphi_X(q, t) = P_X(q) \frac{(1+y^0)^{P_X(q)} + (q^{2m} + \kappa_m - 1)y^0 - q^{2m}t - 1}{1 + (1 - q^{2m} - \kappa_m)y^0 + q^{2m}t}.$$

To see that this is equivalent to (13.24) one can derivate (13.24) in t and use (13.23).

14 Method of VKS-Trees

Let us briefly review the method of VKS-trees [3, 4]. Irreducible class 1 representations of the orthogonal group $O(n)$ are realized in the space of scalar functions

on a unit sphere S_{n-1} in Euclidean space $\mathbb{R}^n$, which are eigenfunctions of the Laplace operator on sphere [3]:

$$\Delta_{\Omega} Y_k(\Omega) + k(k + n - 2)Y_k(\Omega) = 0. \quad (14.1)$$

The method of VKS-trees is based on consequent factorization of Laplacian Δ_{Ω} , for which it is required to introduce polyspherical coordinates.

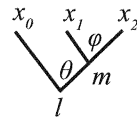
DEFINITION 14.1 *Let $x_0, x_1, \dots, x_{n-1}$ be the Cartesian coordinates of a point on a sphere S_n . We depict them as lines (see Figure 24a). We join the lines in such way that, at first stage, no more than two lines (edges) meet in a vertex and, at the second stage, no more than two previous joined lines meet in a vertex and so on. As a result we obtain a **VKS – tree** (see Figure 24b).*

As can be seen from the figure, there are two kinds of lines: lines with nodes at both ends and lines with one node. The former lines are called internal and the latter free (or dangling) ends. The number of dangling ends is equal to dimension of space. The number of vertices is equal to number of parameters, determining the location of a point on sphere. To each vertex we assign an angle θ_k . The lines outgoing from the vertex θ_k to the left correspond to $\cos \theta_k$ and those outgoing to the right correspond to $\sin \theta_k$. The vertex θ is a vertex of a graph. Then the path, say, from vertex θ to vertex φ_2 (see Figure 24b) can be represented as a product of internal lines, i.e. a product of cosines and sines, which occur along this path

$$h_{\varphi_2} = \prod_{\theta_k=\theta}^{\varphi_2} = \overset{\theta_k}{\bullet} \text{---} \overset{\theta_{k+1}}{\bullet} = \cos \theta \cos \beta \sin \alpha. \quad (14.2)$$

Cartesian coordinates can be determined in terms of angles by multiplying the coefficients h_{φ_k} by the dangling end associated with the angle φ_k .

In three-dimensional space, the following VKS-tree corresponds to the spherical coordinate system



$$\begin{aligned} x_0 &= \cos \theta, \\ x_1 &= \sin \theta \cos \varphi, \\ x_2 &= \sin \theta \sin \varphi. \end{aligned} \quad (14.3)$$

A relation of equivalence can be introduced on the set of VKS-trees: two VKS-trees belong to the same class if one of them can be transformed into the other

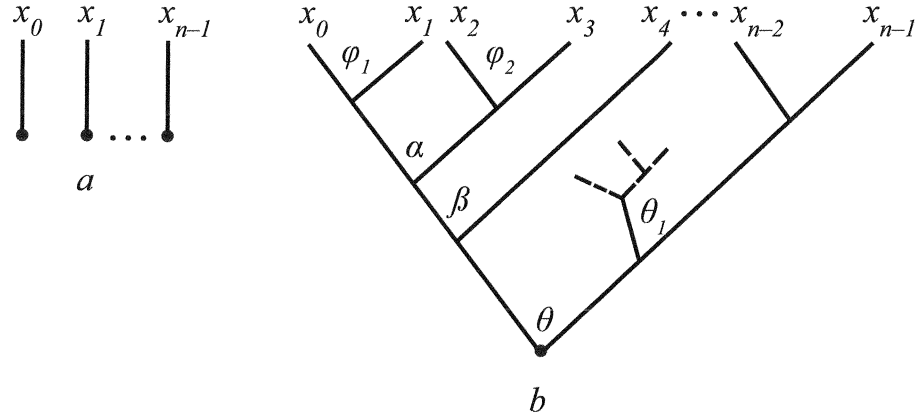


Figure 24: VKS-tree.

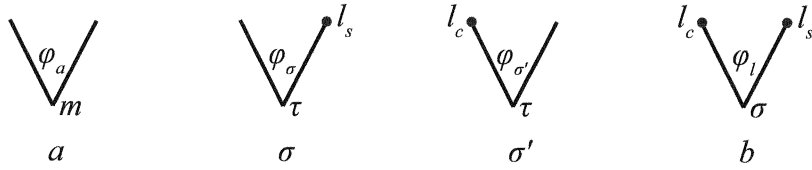


Figure 25: Elementary cells.

by rotation around vertical (perpendicular to the plane of the diagram) axis (or axes), passing through vertex (or vertices). This relation of equivalence enables to split the set of VKS-trees into classes.

VKS-trees belonging to different classes, are truly different structures, i.e. truly topologically inequivalent constructions. There is one class of VKS-trees (systems of polyspherical coordinates) in three-dimensional space, two classes in four-dimensional space, and three classes in five-dimensional space.

In his studies of topologically different VKS-trees, G.I. Kuznetsov [68] has given the following inductive algorithm for writing down Laplace operator Δ_Ω : Equation (14.1) for Figure 24b in polyspherical coordinates can be written as

follows

$$\left\{ \frac{1}{\cos^c \theta \sin^s \theta} \frac{\partial}{\partial \theta} \cos^c \theta \sin^s \theta \frac{\partial}{\partial \theta} + \frac{\Delta_{\Omega_c}(\beta)}{\cos^2 \theta} + \frac{\Delta_{\Omega_s}(\theta_1)}{\sin^2 \theta} + k(k+n-1) \right\} Y_k(\Omega) = 0, \quad (14.4)$$

where, c is number of subsequent vertices to the left of vertex θ ; s is number of subsequent vertices to the right of vertex θ , and $c + s = n - 2$.

Laplacians $\Delta_{\Omega_c}(\beta)$ and $\Delta_{\Omega_s}(\theta_1)$ are given, respectively, on c - and s -dimensional spheres. For $\Delta_{\Omega_c}(\beta)$ and $\Delta_{\Omega_s}(\theta_1)$, the VKS-trees have left and right parts, counting from the vertex θ of initial VKS-tree, and the roots are β and θ_1 . The algorithm for writing these Laplacians in polyspherical coordinates remains the same as for Δ_{Ω} . Constructing the Laplacian Δ_{Ω} in this way, gives its form in polyspherical coordinates.

The equation (14.4) can be solved by the method of variable separation. This yields a constant at each vertex. The constant the associated angle endow the vertex with additional characteristics. To solve equation (14.4), first solve the partial differential equation for each vertex. The complete solution given by the product of the partial solutions. The three kinds of vertices (cells) which occur with this approach are shown in Figure 25. The requirement of one-fold covering of the sphere imposes the following restrictions on the range of angles [3]:

$$0 \leq \varphi_a \leq 2\pi, \quad 0 \leq \varphi_{\sigma} \leq \pi, \quad -\frac{\pi}{2} \leq \varphi_{\sigma'} \leq \frac{\pi}{2}, \quad 0 \leq \varphi_l \leq \frac{\pi}{2}. \quad (14.5)$$

Let us consider an elementary cell of the graph shown at Figure 25b. Here m , τ , l_c , l_s , σ are constants of separation of variables. The equation for the variable $\theta = \varphi_l$ is as follows:

$$\left[\frac{1}{\cos^c \theta \sin^s \theta} \frac{\partial}{\partial \theta} \cos^c \theta \sin^s \theta \frac{\partial}{\partial \theta} - \frac{l_c(l_c + c - 1)}{\cos^2 \theta} - \frac{l_s(l_s + s - 1)}{\sin^2 \theta} + \sigma(\sigma + c + s) \right] \Psi(\theta) = 0, \quad (14.6)$$

and its solution can be written as

$$\begin{aligned}\Psi_{n,c,s}^{\alpha,\beta}(\theta) &= N \cos^{l_c} \theta \sin^{l_s} \theta \mathcal{P}_n^{\alpha,\beta}(\cos 2\theta), \\ 2n &= \sigma - l_c - l_s, \quad \alpha = l_s + \frac{s-1}{2} \equiv l_s + \frac{1}{2}S_{l_s}, \\ \beta &= l_c + \frac{c-1}{2} \equiv l_c + \frac{1}{2}S_{l_c}, n = 0, 1, 2, \dots,\end{aligned}\tag{14.7}$$

if and only if $n \geq 0$ i.e. $\sigma \geq l_c + l_s$. Here, $c(s)$ is number of left (right) vertices corresponding to the vertex σ ; S_{l_s} , S_{l_c} are the numbers of vertices above corresponding to vertices l_s and l_c , $\mathcal{P}_n^{\alpha,\beta}$ are Jakobi polynomials or hyperspherical functions, N is normalizing factor.

Let us consider the cell shown in Figure 25 σ . It is worth of noticing that the cell in Figure 25 σ' is the same as that of Figure 25 σ except for one detail: $\cos \varphi_\sigma$ is substituted for $\sin \varphi_\sigma$.

The equation for the variable $\theta = \varphi_\sigma$ is

$$\left[\frac{1}{\sin^s \theta} \frac{\partial}{\partial \theta} \sin^2 \theta \frac{\partial}{\partial \theta} - \frac{l_s(l_s + s - 1)}{\sin^2 \theta} + \tau(\tau + s) \right] \Psi(\theta) = 0, \tag{14.8}$$

and its solution can be written as follows:

$$\begin{aligned}\Psi_{n,0,s}^{\alpha,\alpha}(\theta) &= N_1 \sin^{l_s} \theta \mathcal{P}_n^{\alpha,\alpha}(\cos \theta), \\ 0 \leq \theta \leq \pi, \quad n &= \tau - l_s, \quad \alpha = l_s + \frac{s-1}{2}, n \geq 0, 1, 2, \dots,\end{aligned}\tag{14.9}$$

where $\mathcal{P}_n^{\alpha,\alpha}$ are Gegenbauer polynomials, N_1 is normalizing coefficient.

For the cell shown in Figure 25a, there is the corresponding the obvious solution

$$\Psi_m(\varphi_a) = \frac{1}{\sqrt{2\pi_a}} e^{im\varphi_a}, \quad 0 \leq \varphi_a \leq 2\pi, \quad m \in \mathbb{Z}. \tag{14.10}$$

As an example, let us consider a class 1 representation of the group $O(3)$. In spherical coordinates, the factorized solution of equation (14.1) can be written as follows (see Figure 14.3):

$$Y_{lm}(\theta, \varphi) = \bar{P}_{lm}(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \tag{14.11}$$

where $Y_{lm}(\theta, \varphi)$ is the spherical function which is an eigenfunction of Laplace operator on sphere S_2 .

15 Polyspherical Coordinates in Cayley-Klein Spaces

Cayley-Klein geometries of dimension n are realized on spheres

$$S^n(\mathbf{j}) = \left\{ \mathbf{x} \in \mathbb{R}R^{n+1}(\mathbf{j}) \mid x_0^2 + \sum_{k=1}^n x_k^2 \prod_{m=1}^k j_m^2 = 1 \right\} \quad (15.1)$$

in spaces $\mathbb{R}R^{n+1}(\mathbf{j})$ resulting from the Euclidean space $\mathbb{R}R^{n+1}$ under the mapping [5]

$$\begin{aligned} \psi : \mathbb{R}R^{n+1} &\rightarrow \mathbb{R}R^{n+1}(\mathbf{j}) \equiv \mathbb{R}V_{n+1}(\mathbf{j}), & \psi x_0^* &= x_0, & \psi x_k^* &= x_k \prod_{m=1}^k j_m, \\ \mathbf{j} &= (j_1, \dots, j_n); & j_k &= 1, \iota_k, i, & k &= 1, 2, \dots, n. \end{aligned} \quad (15.2)$$

The combination of all possible values of the parameters $\mathbf{j}$ produces 3^n different real Cayley-Klein spaces $\mathbb{R}R^{n+1}(\mathbf{j}) \equiv \mathbb{R}V_{n+1}(\mathbf{j})$ [5]. A unified description of all 3^n Cayley-Klein geometries (geometries of constant curvature space) can be given as a domain of an n -dimensional spherical space parametrized by “concrete” coordinates³. Here the total number of nonisomorphic geometries is $N + n = [(3 + \sqrt{5})^{n+1} - (3 - \sqrt{5})^{n+1}] / 2^{n+1} \sqrt{5}$ [5]. The operation of some transition between their groups is based on introducing a set of the parameters $(\mathbf{j} := j_1, \dots, j_n)$. Each of the parameters can take on three values: real, purely imaginary and dual units ι [5].

Under the mapping (15.2) transforming Euclidean space $\mathbb{R}R^n$ into Cayley-Klein space $\mathbb{R}R^n(\mathbf{j})$, Cartesian coordinates in $\mathbb{R}R^n$ are multiplied by products of parameters $\mathbf{j}$. In [5, § 5.2–5.5] it is shown that angles are multiplied by some products of parameters $\mathbf{j}$ as well. Under the mapping ψ , the symmetry (“equality”) of Cartesian coordinates disappears. In method of VKS-trees, this reveals is revealed by the fact that the operation of rotation around vertical axis passing through vertex does not transform a VKS-tree into an equivalent VKS-tree; the partition of the set of VKS-trees into classes is not possible. The other peculiarity is that, for imaginary values of the parameters $\mathbf{j}$, the sphere $S_{n-1}(\mathbf{j})$ can not

³By “concrete” coordinates, we mean real, purely imaginary or dual numbers. The latter were introduced by Clifford [69] in such a way that a dual number itself differs from zero but vanishes when squared.

be covered by one (polyspherical) map. For example, the Minkowski plane can be covered by four system of polar coordinates [70]. To simplify the exposition, we shall consider in this paragraph only contractions of groups, i.e. parameters $j_k = 1, \iota_k$, $k = 1, 2, \dots, n-1$.

In the space $\mathbb{R}R_3(\mathbf{j})$, the sphere $S_2(\mathbf{j}) = \{\mathbf{x} \mid x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2 = 1\}$ admits two systems of spherical coordinates:

$$\begin{aligned} x_0 &= \cos j_1 j_2 \theta \cos j_1 \varphi, \\ x_1 &= \frac{1}{j_1} \cos j_1 j_2 \theta \sin j_1 \varphi, \\ x_2 &= \frac{1}{j_1 j_2} \sin j_1 j_2 \theta, \end{aligned} \quad (15.3)$$

$$\varphi \in \Phi(j_1) = \begin{cases} [0, 2\pi], & j_1 = 1, \\ \mathbb{R}, & j_1 = \iota_1, \end{cases} \quad (15.4)$$

$$\theta \in \Theta(j_1 j_2) = \begin{cases} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], & j = 1, \\ \mathbb{R}, & j \neq 1; \end{cases} \quad (15.5)$$

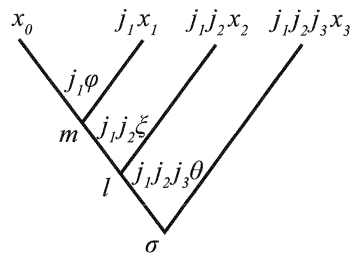
$$\begin{aligned} x_0 &= \cos j_1 \xi, \\ x_1 &= \frac{1}{j_1} \sin j_1 \xi \cos j_2 \alpha, \\ x_2 &= \frac{1}{j_1 j_2} \sin j_1 \xi \sin j_2 \alpha, \end{aligned} \quad (15.6)$$

$$\alpha \in \Phi(j_2), \quad \xi \in \mathbb{Z}(j_1, j_2) = \begin{cases} \mathbb{Z}_0(j_1), & j_2 = 1, \\ \Phi(j_1), & j_2 = \iota_2, \end{cases} \quad (15.7)$$

$$\mathbb{Z}_0(j_1) = \begin{cases} [0, \pi], & j_1 = 1, \\ \mathbb{R}^+, & j_1 = \iota_1. \end{cases} \quad (15.8)$$

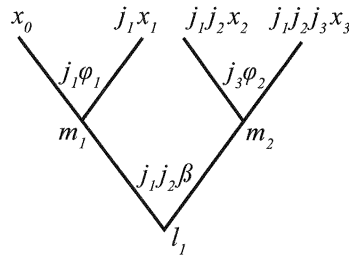
For $j_1 = \iota_1$ both systems of coordinates describe the connected component of sphere.

The sphere $S_3(\mathbf{j}) = \{\mathbf{x} \mid x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2 + j_1^2 j_2^2 j_3^2 x_3^2 = 1\}$ in the space $\mathbb{R}R^4(\mathbf{j})$ admits three systems of polyspherical coordinates:



$$\begin{aligned}
 x_0 &= \cos j_1 \varphi \cos j_1 j_2 \xi \cos j_1 j_2 j_3 \theta, \\
 x_1 &= \frac{1}{j_1} \sin j_1 \varphi \cos j_1 j_2 \xi \cos j_1 j_2 j_3 \theta, \\
 x_2 &= \frac{1}{j_1 j_2} \sin j_1 j_2 \xi \cos j_1 j_2 j_3 \theta, \\
 x_3 &= \frac{1}{j_1 j_2 j_3} \sin j_1 j_2 j_3 \theta,
 \end{aligned}
 \tag{15.9}$$

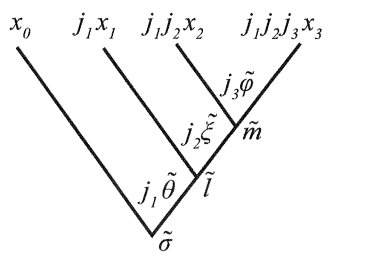
$$\varphi \in \Phi(j_1), \quad \xi \in \Theta(j_1 j_2), \quad \theta \in \Theta(j_1 j_2 j_3) = \begin{cases} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], & \mathbf{j} = \mathbf{1}, \\ \mathbb{R}, & \mathbf{j} \neq \mathbf{1}; \end{cases}$$



$$\begin{aligned}
 x_0 &= \cos j_1 \varphi_1 \cos j_1 j_2 \beta, \\
 x_1 &= \frac{1}{j_1} \sin j_1 \varphi_1 \cos j_1 j_2 \beta, \\
 x_2 &= \frac{1}{j_1 j_2} \cos j_3 \varphi_2 \sin j_1 j_2 \beta, \\
 x_3 &= \frac{1}{j_1 j_2 j_3} \sin j_3 \varphi_2 \sin j_1 j_2 \beta,
 \end{aligned}
 \tag{15.10}$$

$$\begin{aligned}
 \varphi_1 &\in \Phi(j_1), \\
 \varphi_2 &\in \Phi(j_3), \\
 \beta &\in \mathcal{B}(j_1, j_2, j_3) = \begin{cases} Z^+(j_1 j_2), & j_3 = 1, \\ \Theta(j_1 j_2), & j_3 = \iota_3, \end{cases}
 \end{aligned}$$

$$Z^+(j_1 j_2) = \begin{cases} [0, \pi], & \mathbf{j} = \mathbf{1}, \\ \mathbb{R}^+, & \mathbf{j} \neq \mathbf{1}; \end{cases}$$



$$\begin{aligned}
 x_0 &= \cos j_1 \tilde{\theta}, \\
 x_1 &= \frac{1}{j_1} \sin j_1 \tilde{\theta} \cos j_2 \tilde{\xi}, \\
 x_2 &= \frac{1}{j_1 j_2} \sin j_1 \tilde{\theta} \sin j_2 \tilde{\xi} \cos j_3 \tilde{\varphi}, \\
 x_3 &= \frac{1}{j_1 j_2 j_3} \sin j_1 \tilde{\theta} \sin j_2 \tilde{\xi} \sin j_3 \tilde{\varphi},
 \end{aligned}
 \tag{15.11}$$

$$\tilde{\varphi} \in \Phi(j_3), \quad \tilde{\xi} \in Z(j_2, j_3), \quad \tilde{\theta} \in Z(j_1, j_2).$$

For $j_1 = \iota_1$, three systems of polyspherical coordinates describe the connected component $x_0 = 1$ of the sphere $S_3(\iota_1, j_2, j_3)$.

16 Equations for Elementary Cells

Let us begin with the cell shown in Figure 25a. We obtain

$$\begin{array}{c} x_{k-1} \quad j_k x_k \\ \quad \searrow \quad \swarrow \\ \quad j_k \varphi \\ \quad \quad \quad m \end{array} \quad \Psi_m(\varphi, j_k) = N e^{im\varphi}. \quad (16.1)$$

Dependence on the parameter j_k is included in the range of values $\Phi(j_k)$ of the variable φ . The constant of separation is $m \in \mathbb{Z}$ for $j_k = 1$ and $m \in \mathbb{R}$ for $j_k = \iota_k$. The normalizing factor is $N = 1/\sqrt{2\pi}$ for $j_k = 1$ and $N = 1$ for $j_k = \iota_k$. In the latter case, the solution $\Psi_m(\varphi, \iota_k)$ is normalized to a delta-function.

The cell shown in Figure 25σ is transformed into the cell

$$\begin{array}{c} k-1 \quad q \\ \quad \searrow \quad \swarrow \\ \quad j_{k+1} \theta \quad l_s \\ \quad \quad \quad \tau \end{array} \quad \theta \in Z(j_k, j_{k+1}), \quad (16.2)$$

where $k-1$ is the order number of coordinate x_{k-1} connected with the left branch outgoing from vertex τ , q is the order number of the of the last coordinate, connected with node l_s , and s is the number of nodes to the right from vertex τ .

To the cell (16.2), there corresponds the sphere

$$S_{q-k+1}(j_k, \dots, j_q) = \left\{ \mathbf{x} \mid x_{k-1}^2 + j_k^2 x_k^2 + \dots + \left(\prod_{r=k}^q j_r^2 \right) x_q^2 = 1 \right\}$$

and to the vertex l_s - sphere $S_{q-k}(j_{k+1}, \dots, j_q) = \left\{ \mathbf{x} \mid x_k^2 + j_{k+1}^2 x_{k+1}^2 + \dots + \left(\prod_{r=k+1}^q j_r^2 \right) x_q^2 = 1 \right\}$. Under mapping ψ , the Laplacian (or, otherwise, Casimir operator of second order) is transformed according to the rule $\Delta_\theta(j) = \left(\prod_{r=k}^q j_r^2 \right) \Delta_{\theta^*}^* (\rightarrow$

), where the asterisk indicates the quantities entering equations (14.8) which are transformed as follows: $\psi\theta^* = j_{k+1}\theta$, $\tau = \tau^*j_kA$, $l_s = l_s^*A$, and $A = \prod_{r=k+1}^q j_r$.

Transforming (14.8), we obtain the equation

$$\left\{ \frac{A^2}{\sin^2 j_k \theta} \frac{\partial}{\partial \theta} \sin^s j_k \theta \frac{\partial}{\partial \theta} - j_k^2 \frac{l_s[l_s + (s-1)A]}{\sin^2 j_k \theta} + \tau(\tau + sA) \right\} \Psi(\theta) = 0 \quad (16.3)$$

which corresponds to cell (16.2). Its formal solution is the function

$$\Psi_{n,0,s}^{\alpha,\alpha}(\theta) = N(\sin j_k \theta)^{l_s} \mathcal{P}_{\tau-j_k l_s}^{\alpha,\alpha}(\cos j_k \theta), \quad (16.4)$$

$$\alpha = A\alpha^*(\rightarrow) = l_s + \frac{s-1}{2}A, \quad n = j_k A n^*(\rightarrow) = \tau - j_k l_s,$$

where N is a normalizing factor.

Let $j_k = \iota_k$ and $A \neq \iota$, i.e. $A = 1$. Then $\sin \iota_k \theta = \iota_k \theta$, and equation (16.3) turns into

$$\Psi'' + \frac{A_s}{\theta} \Psi' + \left[\tau^2 - \frac{l_s(l_s + (s-1)A)}{\theta^2} \right] \Psi = 0. \quad (16.5)$$

This is the Lommel equation [71], which can be expressed in terms of Bessel functions:

$$\Psi_{\tau,l_s,s}(\theta) = \theta^{\frac{1-s}{2}} J_{l_s + \frac{s-1}{2}}(\tau\theta). \quad (16.6)$$

Let $A = \iota$, i.e. some of the parameters j_r , $k+1 \leq r \leq q$, take on dual values. Then (16.3) can be rewritten as the algebraic equation

$$\left(\tau^2 - j_k^2 \frac{l_s^2}{\sin^2 j_k \theta} \right) \Psi(\theta) = 0, \quad (16.7)$$

which connects the constants of separation in neighbouring vertices

$$l_s = \tau \frac{1}{j_k} \sin j_k \theta. \quad (16.8)$$

The case $j_k = \iota_k$, $A = \iota$ is obtained from (16.8) with $j_k = \iota_k$; the result is $l_s = \tau\theta$. Comparing the formal solution (16.4) with the solution (16.6), we obtain the final relation for Gegenbauer polynomials

$$(\iota_k \theta)^{l_s} \mathcal{P}_{\tau-\iota_k l_s}^{\alpha,\alpha}(\cos \iota_k \theta) = \theta^{\frac{1-s}{2}} J_\alpha(\tau\theta) \quad (16.9)$$

written up to normalizing factors.

The cell shown in Figure 25 σ' is transformed into

$$\begin{array}{c}
 \begin{array}{c} k \\ \bullet \\ l_c \end{array} \begin{array}{c} \backslash \\ A\theta \\ / \end{array} \begin{array}{c} p \\ \bullet \\ \tau \end{array}
 \end{array}
 \quad
 \theta \in \Theta(A) = \begin{cases} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], & A = 1, \\ \mathbb{R}, & A \neq 1, \end{cases}
 \quad (16.10)$$

$$A = \prod_{r=k+1}^p j_r,$$

where k is the order number of the first coordinate x_k connected with vertex l_c , p is the order number of coordinate, connected with the right branch outgoing from vertex τ and c is number of nodes to the left of vertex τ .

To cell (16.10) there corresponds the sphere

$$S_{p-k}(j_{k+1}, \dots, j_p) = \left\{ \mathbf{x} \mid x_k^2 + j_{k+1}^2 x_{k+1}^2 + \dots + \left(\prod_{r=k+1}^p j_r^2 \right) x_p^2 = 1 \right\}$$

and to the vertex l_c , the sphere $S_{p-k-1}(j_{k+1}, \dots, j_{p-1})$. Under mapping ψ , the Laplacian is transformed according to the rule $\Delta_\theta(j) = A^2 \Delta_{\theta^*}^*(\rightarrow)$, where the asterisk indicates the quantities in the equation of the cell shown in Figure 25 σ . These quantities can be written as follows: $\psi\theta^* = A\theta$, $\tau = \tau^*A$, $l_c = l_c^*B$, where $B = \prod_{r=k+1}^{p-1} j_r$, i.e. $A = j_p B$. To cell (16.10), there corresponds the equation

$$\left\{ \frac{1}{\cos^c A\theta} \frac{\partial}{\partial \theta} \cos^c A\theta \frac{\partial}{\partial \theta} - j_p^2 \frac{l_c[l_c + (c-1)B]}{\cos^2 A\theta} + \tau(\tau + cA) \right\} \Psi(\theta) = 0. \quad (16.11)$$

Its formal solution can be written as follows

$$\begin{aligned}
 \Psi_{n,0}^{\alpha,\alpha}(\theta) &= N(\cos A\theta)^{l_c} \mathcal{P}_{\tau-j_p l_s}^{\alpha,\alpha}(\sin A\theta), \\
 \alpha &= B\alpha^*(\rightarrow) = l_c + \frac{c-1}{2}B, \quad n = An^*(\rightarrow) = \tau - j_p l_c.
 \end{aligned}
 \quad (16.12)$$

Let $B = \iota$, some of the parameters j_r , $k+1 \leq r \leq p-1$, are equal to dual units, and $j_p \neq \iota_p$. Then the equation (16.11) takes following form:

$$\Psi''(\theta) + (\tau^2 - j_p^2 l_c^2) \Psi(\theta) = 0, \quad (16.13)$$

and its solution can be written as

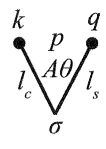
$$\Psi_{\tau, l_c}(\theta) = e^{i\theta\sqrt{\tau^2 - j_p^2 l_c^2}}. \quad (16.14)$$

The case $j_p = l_p$ is obtained from (16.13) and (16.14). Comparing (16.12) and (16.14), we obtain limit relations for Gegenbauer polynomials

$$\begin{aligned} \mathcal{P}_{\tau - j_p l_c}^{\alpha, \alpha}(\sin \iota j_p \theta) &= e^{i\theta\sqrt{\tau^2 - j_p^2 l_c^2}}, \quad \alpha = l_c + \iota \frac{c-1}{2}, \\ \mathcal{P}_{\tau - \iota_p l_c}^{\alpha, \alpha}(\sin \iota_p \theta) &= e^{i\theta\tau}, \quad \alpha = l_c + \frac{c-1}{2} \end{aligned} \quad (16.15)$$

written up to normalizing factors.

The cell, shown at Figure 25b, is transformed into the cell



$$\theta \in \begin{cases} Z^+(A), & j_{p+2} \neq l_{p+2}, \\ \Theta(A), & j_{p+2} = l_{p+2}, \end{cases} \quad A = \prod_{r=k+1}^{p+1} j_r,$$

$$Z^+(A) = \begin{cases} \left[0, \frac{\pi}{2}\right], & A = 1, \\ \mathbb{R}^+, & A \neq 1, \end{cases} \quad \Theta(A) = \begin{cases} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], & A = 1, \\ \mathbb{R}, & A \neq 1, \end{cases} \quad (16.16)$$

where k is order number of the first coordinate x_k , connected with the left vertex l_c ; p is order number of the last coordinate connected with vertex l_c ; q is order number of the last coordinate connected with the right vertex l_s .

To cell (16.16) there corresponds sphere

$$\begin{aligned} S_{q-k}(j_{k+1}, \dots, j_q) &= \left\{ \mathbf{x} | x_k^2 + j_{k+1}^2 x_{k+1}^2 + \dots + \left(\prod_{r=k+1}^p j_r^2 \right) x_p^2 + \right. \\ &\quad \left. + A \left(x_{p+1}^2 + j_{p+2}^2 x_{p+2}^2 + \dots + \left(\prod_{r=p+2}^q j_r^2 \right) x_q^2 \right) = 1 \right\}. \end{aligned}$$

Equating the expression in round brackets to unit, we can obtain sphere $S_{q-p-1}(j_{p+2}, \dots, j_q)$, which corresponds to vertex l_s , and equating to unit the expression in front of round brackets, we are able to write the sphere $S_{p-k}(j_{k+1}, \dots, j_p)$, corresponding to vertex l_c . Under mapping ψ Laplacian (14.6) is transformed according to the rule $\Delta_\theta(j) = \left(\prod_{r=k+1}^q j_r^2 \right) \Delta_{\theta^*}(\rightarrow)$, and quantities, entering (14.6),

are transformed as follows:

$$\psi\theta^* = A\theta, \sigma = \sigma^* \prod_{r=k+1}^q j_r, l_c = l_c^* A/j_{p+1}, l_s = l_s^* B, \text{ where } B = \prod_{r=p+2}^q j_r.$$

Transforming (14.6), we obtain equation, corresponding to cell (16.16):

$$\left\{ \frac{B^2}{\cos^c A\theta \sin^s A\theta} \frac{\partial}{\partial \theta} \cos^c A\theta \sin^s A\theta \frac{\partial}{\partial \theta} - j_{p+1}^2 B^2 \frac{l_c[l_c + (c-1)A/j_{p+1}]}{\cos^2 A\theta} - \right. \\ \left. - A^2 \frac{l_s[l_s + (s-1)B]}{\sin^2 A\theta} + \sigma[\sigma + (c+s)AB] \right\} \Psi(\theta) = 0. \quad (16.17)$$

Its formal solution can be written as follows:

$$\Psi_{n,l_c,l_s}^{\alpha,\beta}(\theta) = N(\sin A\theta)^{l_s} (\cos A\theta)^{l_c} \mathcal{P}_n^{\alpha,\beta}(\cos 2A\theta), \quad (16.18) \\ 2n = \sigma - l_c B j_{p+1} - l_s A, \quad \alpha = l_s + \frac{s-1}{2} B, \quad \beta = l_c + \frac{c-1}{2 j_{p+1}} A.$$

For $A = 1$, $B = \iota$, i.e. when one or more parameters j_r , $p+2 \leq r \leq q$, take dual values, equation (16.17) is transformed into algebraic equation, connecting constants of separation of variables in neighbouring vertices by relation

$$l_s^2 = \sigma^2 \frac{1}{A^2} \sin^2 A\theta. \quad (16.19)$$

For $B = 1$, $A = \iota$, $j_{p+1} \neq \iota_{p+1}$, equation (16.17) takes form

$$\Psi'' + \frac{s}{\theta} \Psi' + \left[\sigma^2 - j_{p+1}^2 l_c^2 - \frac{l_s(l_s + s - 1)}{\theta^2} \right] \Psi = 0. \quad (16.20)$$

This is Lommel equation [71], which solution can be expressed in terms of Bessel function

$$\Psi_{\sigma,l_s,l_c,s}(\theta) = N \theta^{\frac{l_s-s}{2}} J_{l_s+\frac{s-1}{2}} \left(\theta \sqrt{\sigma^2 - j_{p+1}^2 l_c^2} \right). \quad (16.21)$$

For $j_{p+1} = \iota_{p+1}$ equation and its solution come out of (16.20) and (16.21).

17 Representations of Groups $SO(3; j)$, $SO(4; j)$

Let us consider representations of class 1 of groups $SO(3; j)$, $j_1 = 1, \iota_1$; $j_2 = 1, \iota_2$ in spherical system of coordinates, corresponding to the VKS-tree (15.3). Metrics

on sphere $S_2(\mathbf{j})$ in these coordinates is as follows

$$dl^2(\mathbf{j}) = d\varphi^2 \cos^2 j_1 j_2 \theta + j_2^2 d\theta^2. \quad (17.1)$$

Laplacian on sphere, corresponding to this metrics, is

$$\Delta_3(\mathbf{j}) = \frac{1}{\cos j_1 j_2 \theta} \frac{\partial}{\partial \theta} \cos j_1 j_2 \theta \frac{\partial}{\partial \theta} + \frac{j_2^2}{\cos^2 j_1 j_2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (17.2)$$

Factorized eigenfunctions of Laplacian (17.2), which are solutions of equation $\Delta_3(\mathbf{j}) \Psi(\theta, \varphi) = -\tau(\tau+1) \Psi(\theta, \varphi)$, can be found, multiplying solution (16.2) by solution (16.12) for $c = 1$, $l_c \equiv m$:

$$\Psi_{\tau, m}(\theta, \varphi) = N e^{im\varphi} (\cos j_1 j_2 \theta)^m \mathcal{P}_{\tau-j_2 m}^{m, m}(\sin j_1 j_2 \theta). \quad (17.3)$$

In coordinate system, corresponding to the VKS-tree (15.6), metrics on sphere $S_2(\mathbf{j})$ can be written

$$dl^2(\mathbf{j}) = d\zeta^2 + j_2^2 \frac{1}{j_1^2} \sin^2 j_1 \zeta d\alpha^2, \quad (17.4)$$

and Laplacian takes the following form

$$\Delta_3(\mathbf{j}) = \frac{j_2^2}{\sin j_1 \zeta} \frac{\partial}{\partial \zeta} \sin j_1 \zeta \frac{\partial}{\partial \zeta} + \frac{j_1^2}{\sin^2 j_1 \zeta} \frac{\partial^2}{\partial \alpha^2}. \quad (17.5)$$

Its eigenfunctions can be found, multiplying solution (16.1) and (16.4) for $s = 1$, $l_s \equiv l$, $k = 1$, $q = 2$, which gives

$$\Psi_{\tau, l}(\zeta, \alpha) = N e^{im\alpha} (\sin j_1 \zeta)^l \mathcal{P}_{\tau-j_1 l}^{l, l}(\cos j_1 \zeta). \quad (17.6)$$

On sphere $S_3(\mathbf{j})$ metrics in polyspherical coordinates, corresponding to the VKS-tree (15.9), is given by expression

$$dl^2(\mathbf{j}) = \cos^2 j_1 j_2 j_3 \theta (\cos^2 j_1 j_2 \xi d\varphi^2 + j_2^2 d\xi^2) + j_2^2 j_3^2 d\theta^2. \quad (17.7)$$

It corresponds to Laplacian

$$\Delta_4(\mathbf{j}) = \frac{1}{\cos^2 j_1 j_2 j_3 \theta} \left(\frac{\partial}{\partial \theta} \cos^2 j_1 j_2 j_3 \theta \frac{\partial}{\partial \theta} + \frac{j_3^2}{\cos j_1 j_2 \xi} \frac{\partial}{\partial \xi} \cos j_1 j_2 \xi \frac{\partial}{\partial \xi} + \frac{j_2^2 j_3^2}{\cos j_1 j_2 \xi} \frac{\partial^2}{\partial \varphi^2} \right), \quad (17.8)$$

which eigenfunctions, realizing the representation of class 1 of group $SO(4; \mathbf{j})$, can be found by multiplying functions (16.12) and have the form

$$\begin{aligned} \Psi_{\sigma, l, m}(\theta, \xi, \varphi) &= N e^{im\varphi} (\cos j_1 j_2 \xi)^m (\cos j_1 j_2 j_3 \theta)^l \times \\ &\times \mathcal{P}_{l-j_2 m}^{m, m}(\sin j_1 j_2 \xi) \mathcal{P}_{\sigma-j_3 l}^{l+\frac{1}{2}j_1 j_2, l+\frac{1}{2}j_1 j_2}(\sin j_1 j_2 j_3 \theta). \end{aligned} \quad (17.9)$$

In polyspherical coordinates (15.10) metrics on sphere $S_3(\mathbf{j})$ is as follows:

$$dl^2(\mathbf{j}) = j_2^2 d\beta^2 + \cos^2 j_1 j_2 \beta d\varphi_1^2 + j_3^2 \frac{1}{j_1^2} \sin^2 j_1 j_2 \beta d\varphi_2^2. \quad (17.10)$$

To this metrics there corresponds Laplacian

$$\begin{aligned} \Delta_4(\mathbf{j}) &= \frac{j_3^2}{\sin j_1 j_2 \beta \cos j_1 j_2 \beta} \frac{\partial}{\partial \beta} \sin j_1 j_2 \beta \cos j_1 j_2 \beta \frac{\partial}{\partial \beta} + \\ &+ \frac{j_2^2 j_3^2}{\cos^2 j_1 j_2 \beta} \frac{\partial^2}{\partial \varphi_1^2} + \frac{j_1^2 j_2^2}{\sin^2 j_1 j_2 \beta} \frac{\partial^2}{\partial \varphi_2^2}, \end{aligned} \quad (17.11)$$

and its eigenfunctions, realizing representation of class 1 of group $SO(4; \mathbf{j})$ in coordinate system (15.10), can be found by using (16.1) and (16.18).

These functions are

$$\begin{aligned} \Psi_{l_1, m_1, m_2}(\beta, \varphi_1, \varphi_2) &= \\ &= N e^{im_1 \varphi_1 + im_2 \varphi_2} (\cos j_1 j_2 \beta)^{m_1} (\sin j_1 j_2 \beta)^{m_2} \mathcal{P}_{l_1 - j_2 j_3 m_1 - j_1 j_2 m_2}^{m_2, m_1}(\cos 2j_1 j_2 \beta). \end{aligned} \quad (17.12)$$

Finally, in coordinates (15.11) metrics on sphere $S_3(\mathbf{j})$ can be written as

$$dl^2(\mathbf{j}) = d\tilde{\theta}^2 + \frac{1}{j_1^2} \sin^2 j_1 \tilde{\theta} (j_2^2 d\tilde{\xi}^2 + j_3^2 \sin^2 j_2 \tilde{\xi} d\tilde{\varphi}^2). \quad (17.13)$$

To this metrics there corresponds Laplace operator

$$\begin{aligned} \Delta_4(\mathbf{j}) &= \frac{1}{\sin^2 j_1 \tilde{\theta}} \left(j_2^2 j_3^2 \frac{\partial}{\partial \tilde{\theta}} \sin^2 j_1 \tilde{\theta} \frac{\partial}{\partial \tilde{\theta}} + \right. \\ &\left. + \frac{j_1^2 j_3^2}{\sin j_2 \tilde{\xi}} \frac{\partial}{\partial \tilde{\xi}} \sin j_2 \tilde{\xi} \frac{\partial}{\partial \tilde{\xi}} + \frac{j_1^2 j_2^2}{\sin^2 j_1 \tilde{\xi}} \frac{\partial^2}{\partial \tilde{\varphi}^2} \right), \end{aligned} \quad (17.14)$$

its eigenfunctions

$$\begin{aligned} \Psi_{\tilde{\sigma}, \tilde{l}, \tilde{m}}(\tilde{\theta}, \tilde{\xi}, \tilde{\varphi}) &= N e^{i \tilde{m} \tilde{\varphi}} (\sin j_2 \tilde{\xi})^{\tilde{m}} (\sin j_1 \tilde{\theta})^{\tilde{l}} \times \\ &\times \mathcal{P}_{\tilde{l}-j_2 \tilde{m}}^{\tilde{m}, \tilde{m}}(\cos j_2 \tilde{\xi}) \mathcal{P}_{\tilde{\sigma}-j_1 \tilde{l}}^{\tilde{l}+\frac{1}{2} j_2 j_3, \tilde{l}+\frac{1}{2} j_2 j_3}(\cos j_1 \tilde{\theta}) \end{aligned} \quad (17.15)$$

can be found, using the described algorithm. They realize representation of class 1 of group $SO(4; \mathbf{j})$ in polyspherical coordinate system (15.11). Using relations from § 16, it is easy to find eigenfunctions of Laplace operators of contracted groups [72–76].

18 Functor category of VKS-trees

Let us briefly review a category **Euclid** and classical Cayley-Klein categories [5, § 7.1.5].

A Euclidean vector space $\mathbb{R}R^{n+1}$ is a vector space over the field $\mathbb{R}$ of real numbers-equipped with an inner product function $(,) : \mathbb{R}R^{n+1} \times \mathbb{R}R^{n+1} \rightarrow \mathbb{R}$ which is bilinear, symmetric, and positive definite. These spaces are the objects of a category **Euclid**, with morphisms those linear maps which preserve inner product. There are two functors

$$U : \mathbf{Euclid} \rightarrow \mathbf{Vct}_{\mathbb{R}}, \quad * : (\mathbf{Euclid})^{op} \rightarrow \mathbf{Vct}_{\mathbb{R}}$$

to the category of real vector spaces: The (covariant) forgetful functor U “forget the inner product” and the contravariant functor “take the dual space”.

On the other hand, in the previous Section we have found representations of orthogonal groups in Cayley-Klein spaces and we have shown that their Laplace (Casimir) operators on sphere and other algebraic constructions can be obtained by transferring the corresponding constructions for classical Lie groups in accord with (15.2). Such an approach is natural and justified by the fact that classical Lie groups and their characteristic algebraic constructions are well studied. But is such an approach the only one? Is it possible to take as the initial space not only the Euclidean space but also a Cayley-Klein space? The positive answer to this question was given by Theorem on the structure of transitions between Cayley-Klein spaces [77, 78].

The transitions from the $(n+1)$ -dimensional real Cayley-Klein space $\mathbb{R}V^{n+1}(\mathbf{j})$ to the real Cayley-Klein space $\mathbb{R}V^{n+1}(\mathbf{j}')$, and from the groups $SO(n+1, \mathbb{R}; \mathbf{j})$,

$Sp(n, \mathbb{R}; \mathbf{j})$ to the groups $SO(n+1, \mathbb{R}; \mathbf{j}')$, $Sp(n, \mathbb{R}; \mathbf{j}')$ as well, can be, respectively, obtained from (15.2) and the transitions

$$\psi : \mathbb{R}V^{n+1}(\mathbf{j}) \rightarrow \mathbb{R}V^{n+1}(\mathbf{j}'), \quad \psi'x_0 = x'_0, \quad \psi'x_k = x'_k \prod_{m=1}^k j'_m j_m^{-1}, \quad (18.1)$$

by the same substitution of parameters j_k for $j'_k j_k^{-1}$, where $\mathbf{j} = (j_1, \dots, j_n)$ and each of parameters j_k assuming three values: $j_k = 1, \iota_k, i$.

Similarly the permissibility of these transitions can be justified for complex Cayley-Klein space $\mathbb{C}V^{n+1}(\mathbf{j})$ which emerge from the $(n+1)$ -dimensional complex Euclidean space $\mathbb{C}R^{n+1}$ by the mapping

$$\begin{aligned} \psi : \mathbb{C}R^{n+1} &\rightarrow \mathbb{C}V^{n+1}(\mathbf{j}), \quad \psi z_0^* = z_0, \\ \psi z_k^* &= z_k \prod_{m=1}^k j_m, \quad k = 1, 2, \dots, n, \end{aligned} \quad (18.2)$$

where $z_0^*, z_k^* \in \mathbb{C}R^{n+1}$, $z_0, z_k \in \mathbb{C}V^{n+1}(\mathbf{j})$ are complex Cartesian coordinates. The totality of all possible values of the parameter $\mathbf{j}$ gives 3^n different real $\mathbb{R}V^{n+1}(\mathbf{j})$ and, correspondingly, complex $\mathbb{C}V^{n+1}(\mathbf{j})$ Cayley-Klein spaces.

The quadratic form $(\mathbf{z}^*, \mathbf{z}^*) = \sum_{m=0}^n |z_m^*|^2$ of the space $\mathbb{C}R^{n+1}$ turns into the quadratic form

$$(\mathbf{z}, \mathbf{z}) = |z_0|^2 = \sum_{k=1}^n |z_k|^2 \prod_{m=1}^k j_m^2. \quad (18.3)$$

of the space $\mathbb{C}R^{n+1}(\mathbf{j})$ under the mapping (18.2). Here $|z_k| = (x_k^2 + y_k^2)^{1/2}$ is the absolute value (modulus) of the complex number $z_k = x_k + iy_k$, and $\mathbf{z}$ is the complex vector: $\mathbf{z} = (z_0, z_1, \dots, z_n)$.

Let us define (formally) the transition from the space $\mathbb{C}V^{n+1}(\mathbf{j})$ and generators to the space $\mathbb{C}V^{n+1}(\mathbf{j}')$ by transformations, which can be obtained from the transformations (18.2), substituting in the latter the parameters j_k for $j'_k j_k^{-1}$, i.e.

$$\begin{aligned} \psi' : \mathbb{C}V^{n+1}(\mathbf{j}) &\rightarrow \mathbb{C}V^{n+1}(\mathbf{j}'), \quad \psi'z_0 = z'_0, \\ \psi'z_k &= z'_k \sum_{m=1}^k j'_m j_m^{-1}, \quad k = 1, 2, \dots, n. \end{aligned} \quad (18.4)$$

A Cayley-Klein space $\mathbb{K}(\mathbf{j}) := \mathbb{R}V^{n+1}(\mathbf{j}), \mathbb{C}V^{n+1}(\mathbf{j}), \mathbb{R}V^n(\mathbf{j}) \otimes \mathbb{R}V^n(\mathbf{j})$ is called non-fiber space, if none of the parameters $j_1, \dots, j_n$ assumes a dual value. A space $\mathbb{K}(\mathbf{j})$ is called $(k_1, k_2, \dots, k_p)$ -fiber space, if $1 \leq k_1 < k_2 < \dots < k_p \leq n$ and $j_{k_1} = \iota_{k_1}, \dots, j_{k_p} = \iota_{k_p}$, and the other $j_k = 1, i$.

However, transitions (18.1) and (18.4) do not make sense for all Cayley-Klein groups and spaces, because for the dual values of parameters $\mathbf{j}$ the expressions $\iota_k^{-1}, \iota_m \iota_k^{-1}$ for $k \neq m$ are not defined. We have defined (see [5]) only expressions $\iota_k \iota_k^{-1}, k = 1, 2, \dots, n$. So if for some k we put $j_k = \iota_k$, then the transformations (18.1) and (18.4) will be defined only in the case when the dashed parameter with the same number is equal to the same purely dual number, i.e. $j'_k = \iota_k$.

These Cayley-Klein spaces are the objects of a Cayley-Klein category $\mathbf{CK}$, with morphisms (18.1) or (18.4) which preserve quadratic form.

Let us introduce the notations: $TV^{n+1}(\mathbf{j})$ for VKS-trees in Cayley-Klein spaces $\mathbb{K}(\mathbf{j})$.

Given Cayley-Klein categories $\mathbf{CKC}$ and $\mathbf{CKB}$, we consider all VKS-tree functors $RV^{n+1}(\mathbf{j}), SV^{n+1}(\mathbf{j}), TV^{n+1}(\mathbf{j}), \dots : \mathbf{CKC} \rightarrow \mathbf{CKB}$. If $\sigma : R \rightarrow S$ and $\tau : S \rightarrow T$, are two natural transformations, their components for each $c \in C$ define composite morphisms $(\tau \cdot \sigma)c \rightarrow \tau c \circ \sigma c$ which are the components of a transformation $\tau \cdot \sigma \rightarrow T$. To show $\tau \cdot \sigma$ natural, take any $f : c \rightarrow c'$ in C and consider the diagram

$$\begin{array}{ccc}
 & Rc & \xrightarrow{Rf} Rc' \\
 \sigma c \downarrow & & \downarrow \sigma c' \\
 (\tau \cdot \sigma)c & Sc & \xrightarrow{Sf} Sc' \\
 \tau c \downarrow & & \downarrow \tau c' \\
 & Tc & \xrightarrow{Tf} Tc' \\
 & & (\tau \cdot \sigma)c'
 \end{array}$$

Since σ and τ are natural, both small squares are commutative. Hence the rectangle commutes, so the composite $\tau \cdot \sigma$ is natural.

This composition of transformations is associative; moreover it has for each functor T an identity, the natural transformation $1_T : T \rightarrow T$ with components $1_T c = 1_{Tc}$. Hence, given the Cayley-Klein categories $\mathbf{CKB}$ and $\mathbf{CKC}$, we may construct formally a *functor category of VKS-trees* $\mathbf{VKS|Tree}$ with objects the functors $R, S, T : \mathbf{CKC} \rightarrow \mathbf{CKB}$ and morphisms the natural transformations between two such functors.

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